RECENT CALCULATIONS IN QCD: $gg \to ZZ$ AT NLO, TOWARDS N3LO DGLAP

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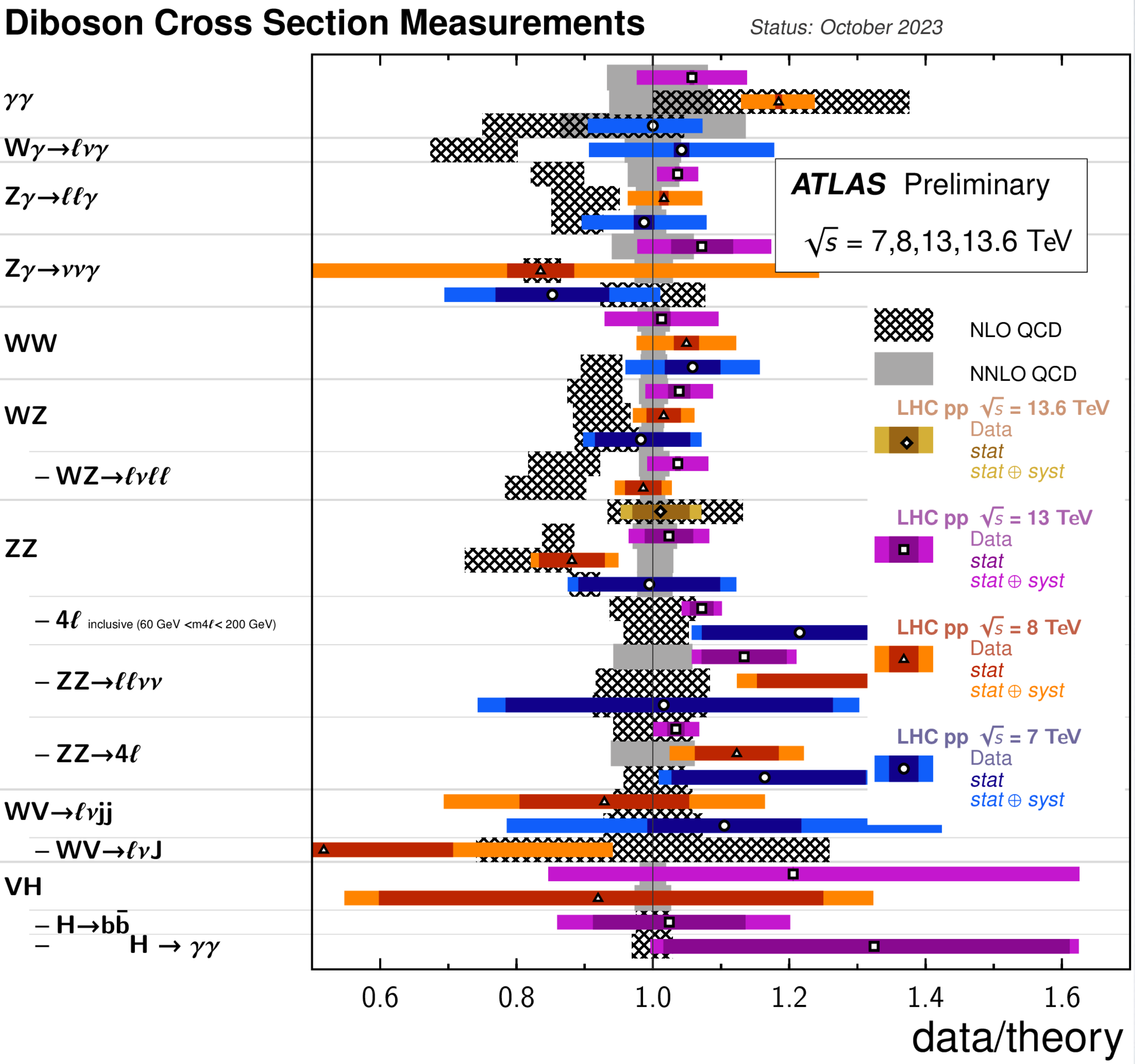
Ringberg 2024: 2nd Workshop on Tools for High Precision LHC Simulations

Amplitude overview talks:
Fabrizio Caola, Federico Buccioni, Vasily Sotnikov

here: focus on two specific calculations

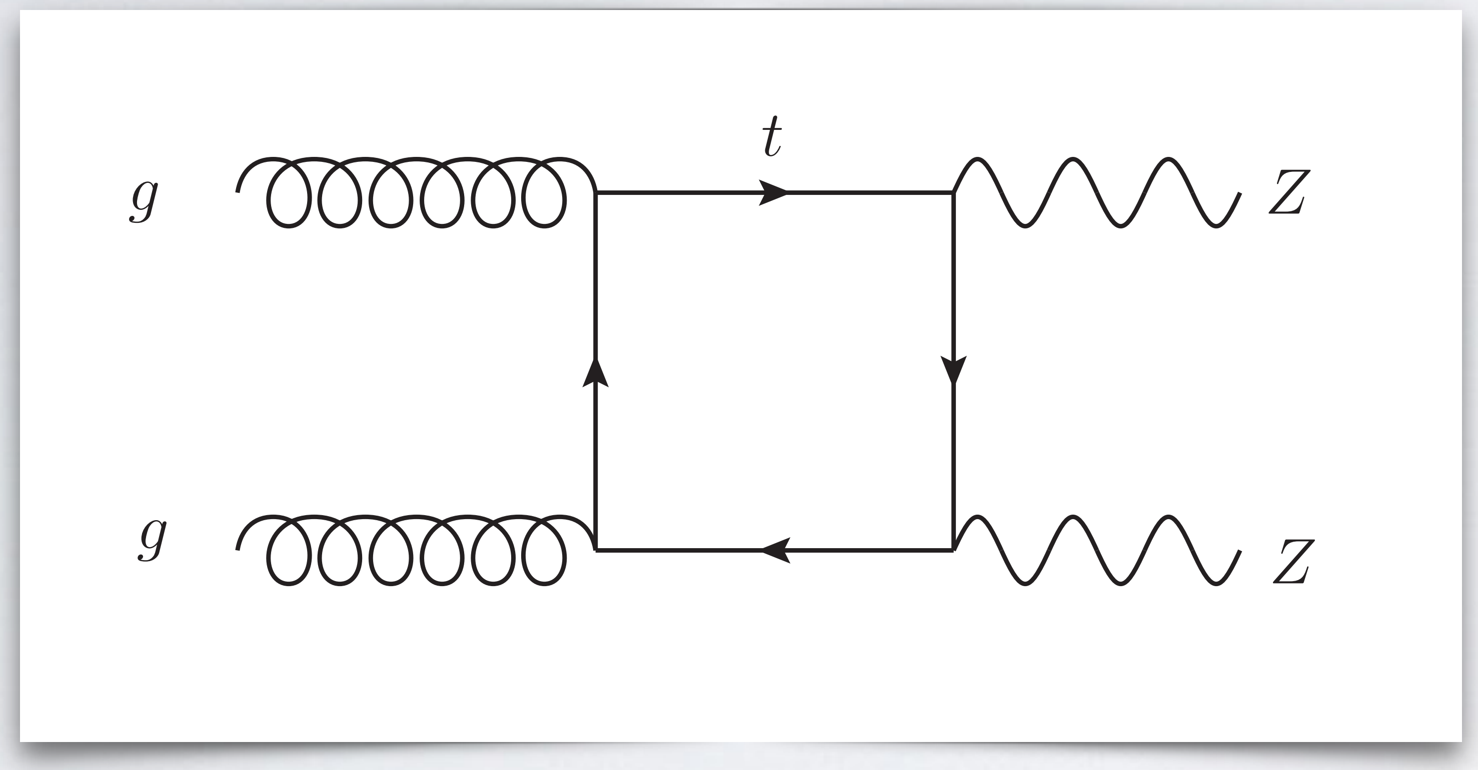
$gg \to ZZ$ AT NLO



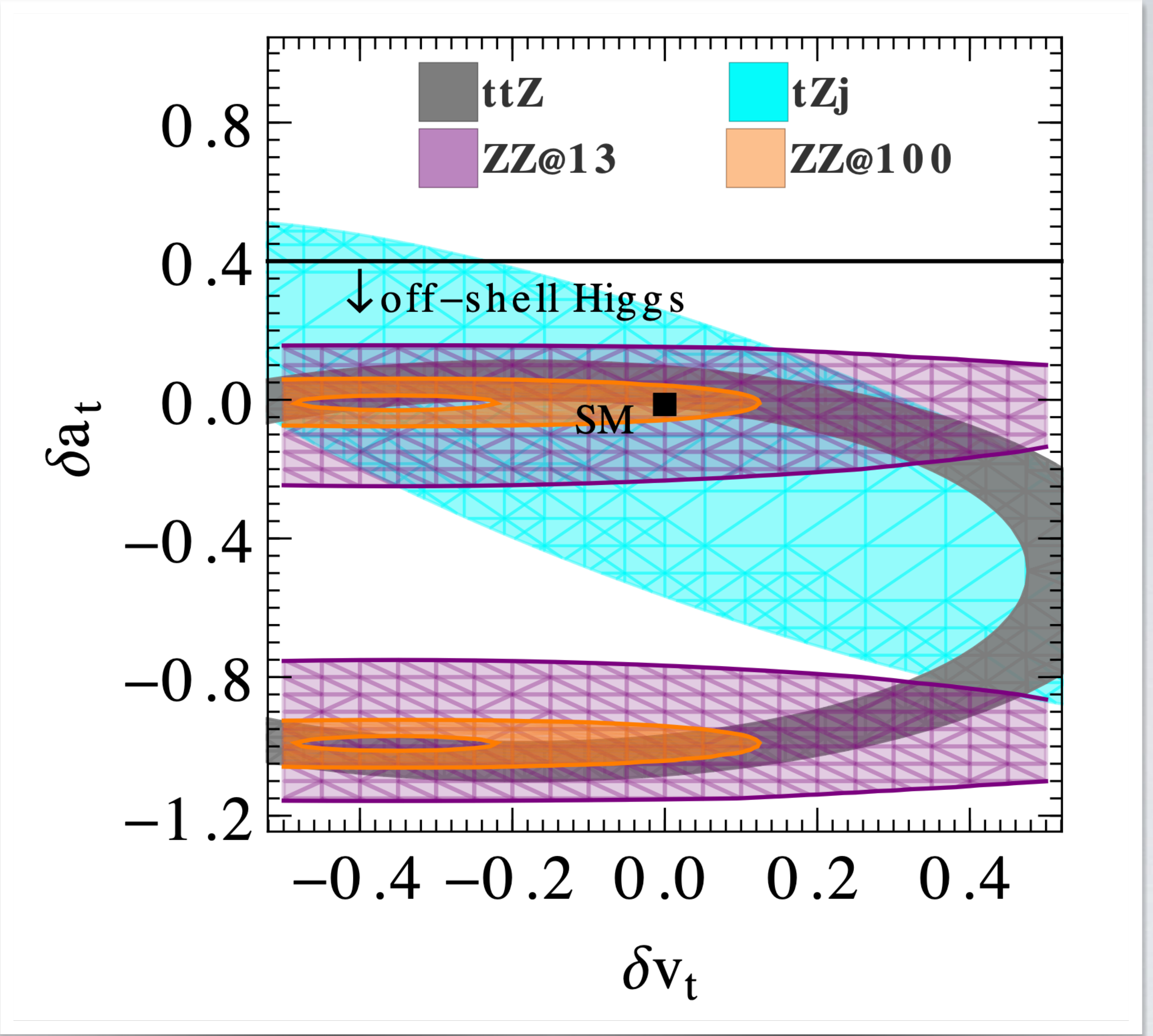
talk: Stefan Kallweit

LONGITUDINAL ZZ PRODUCTION

- **Continuum** ZZ irred. bckgrd. to $H \to ZZ$
- **On-/off-shell Higgs** constrains *width* *[Caola, Melnikov 2013]*
- Goldstone equivalence: longitudinal Z becomes scalar at high E, couplings to t enhanced



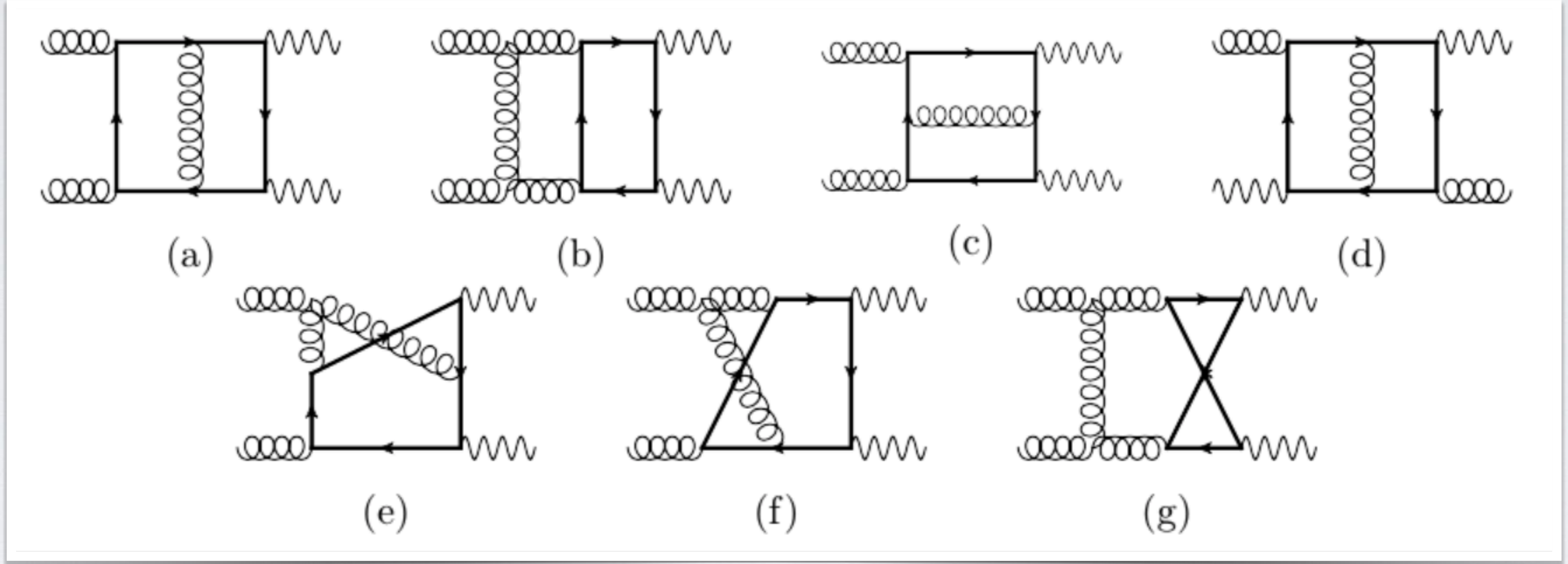
- *ZZ* sensitive to **anomalous axial Ztt** coupling
- Motivates **full** $gg \to ZZ$ **at NLO:** N3LO corrections to hadronic cross sec
- Partial results: *[Melnikov, Dowling '15, Caola, Dowling, Melnikov Röntsch, Tancredi '16, Campbell, Ellis, Czakon, Kirchner '16, Gröber, Maier, Rauth '19, Davies, Mishima, Steinhauser, Wellmann '20]*



[Cao, Yan, Yuan, Zhang 2020]

TOP-QUARKS IN $gg \to ZZ$ AT 2 LOOPS

- Collaboration with Bakul Agarwal and Stephen P. Jones:
  - full 2-loop contributions to $gg \to ZZ$
  - exact top quark mass dependence



INTEGRATION-BY-PART (IBP) IDENTITIES

- IBP identities in dimensional regularization since integrals over total derivatives vanish:

$$\int d^d k_1 \cdots d^d k_L \, \frac{\partial}{\partial k_i^\mu} \left(v^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right) = 0 \, , \quad D_j = q_j^2 - m_j^2 + i\delta \, , \quad v^\mu \text{ loop or ext. mom.}$$

- Implies linear relations between loop integrals *[Chetyrkin, Tkachov ’81]*
- Integer indices: linear system of equations, allows for systematic reduction *[Laporta '00]*
- Only finite number of integrals linearly independent: basis or master integrals

FINITE FIELDS AND RATIONAL RECONSTRUCTION

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients

$$
\begin{array}{ccccccccc}
I_{\mathbb{Z}_p[x]} & \xrightarrow[\text{by number } x_i]{\text{sample } x} & I_{\mathbb{Z}_p, x_1} & \xrightarrow[\text{reduction}]{\text{row}} & O_{\mathbb{Z}_p, x_1} & \xrightarrow[\text{interpolation}]{\text{polynomial}} & O_{\mathbb{Z}_p[x]} & \xrightarrow[\text{reconstruction}]{\text{rational function}} & O_{\mathbb{Z}_p[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_p, x_2} & \longrightarrow & O_{\mathbb{Z}_p, x_2} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_p, x3} & \longrightarrow & O_{\mathbb{Z}_p, x3} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

[AvM, Schabinger ’14; Peraro ’16; …], note: parallelizable, multivariate e.g. by iteration

Syzygy Based IBPs Without Dots

[Gluza, Kajda, Kosower ‘11; Schabinger ’11; Ita ’15; Larsen, Zhang ’15; Böhm, Georgoudis, Larsen, Schulze, Zhang ’18; Agarwal, Jones, AvM ’20, …]

Baikov's parametric representation of Feynman integrals:

$$I(\nu_1, \ldots, \nu_N) = \mathcal{N} \int \mathrm{d}z_1 \cdots \mathrm{d}z_m P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_N^{\nu_N}}$$

[Böhm, Georgoudis, Larsen, Schulze, Zhang ’18]: useful for IBPs without dots

$$0 = \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \sum_{i=1}^{m} \frac{\partial}{\partial z_i} \left(a_i P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_N^{\nu_N}} \right)$$

$$= \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \sum_{i=1}^{N} \left(\frac{\partial a_i}{\partial z_i} + \frac{d-L-E-1}{2P} a_i \frac{\partial P}{\partial z_i} - \frac{\nu_i a_i}{z_i} \right) P^{\frac{d-L-E-1}{2}} \frac{1}{z_1^{\nu_1} \cdots z_N^{\nu_N}}$$

explicit solutions to constraint:

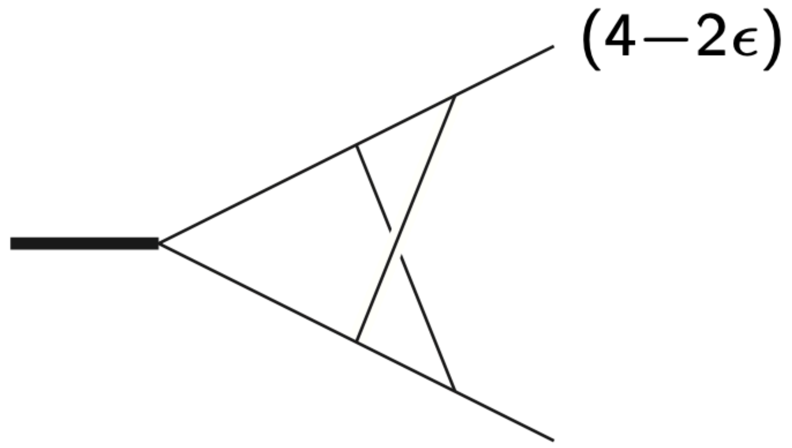
$$\left(\sum_{i=1}^{N} a_i \frac{\partial P}{\partial z_i} \right) + bP = 0 \qquad \text{(absence of dim. shifts)}$$

in addition, require for denominators of sector:

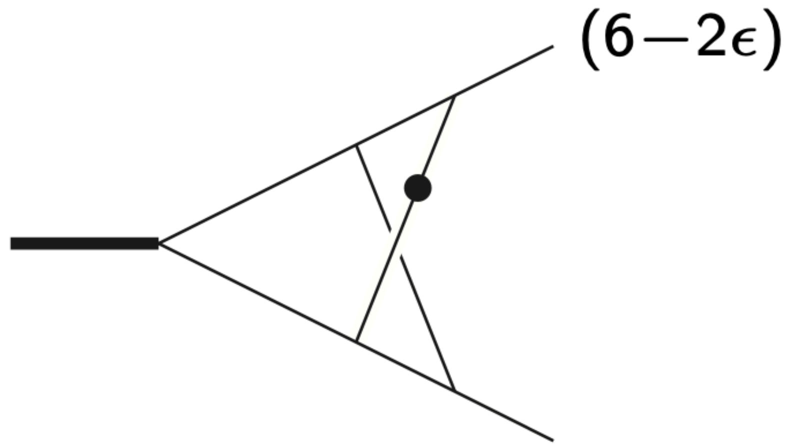
$$a_i = b_i z_i \qquad \text{(absence of dots)}$$

need intersection of two syzygy modules

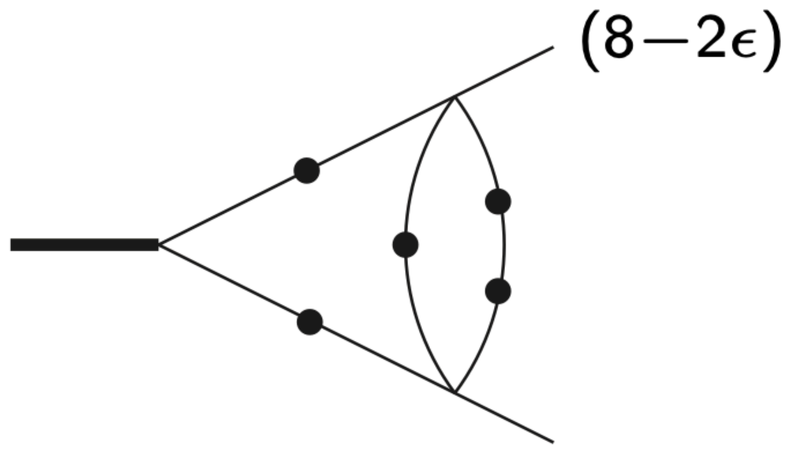
SOLVE INTEGRALS: PARAMETRIC INTEGRATION OF FINITE INTEGRALS



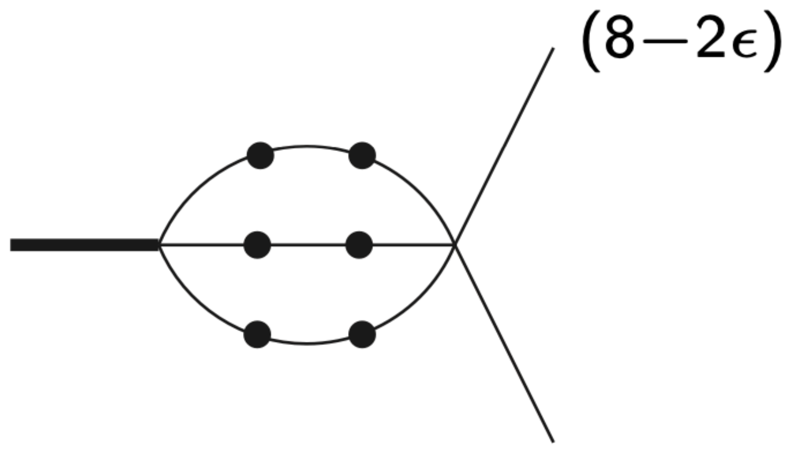
$$= -\frac{4(1-4\epsilon)}{\epsilon(1-\epsilon)q^2}$$



$$-\frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4(1-\epsilon)^2(2-\epsilon)^2q^2}$$



$$+\frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)(1-\epsilon)^2(2-\epsilon)^2q^2}$$

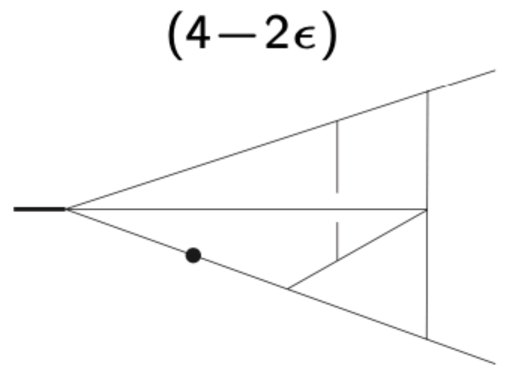


- General observation *[Panzer 2014; AvM, Panzer, Schabinger 2014]*:
  - any ***divergent*** loop integral can be expressed via **finite** basis integrals
  - Reduze 2 finds finite integrals

- Expand integrands of ***finite*** integrals around $\epsilon = (4-d)/2 \approx 0$
  - If linearly reducible: integrate ***analytically*** with HyperInt *[Panzer 2014]*
  - Improved ***numerical*** evaluations, used for HH *[Borowka, Greiner, Heinrich, Jones, Kerner '16]*, …

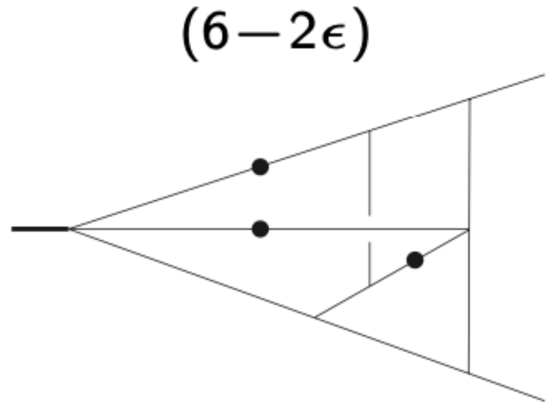
"NICE" FINITE INTEGRALS

- Example: 10 terms in ϵ for weight 6 in conventional basis:



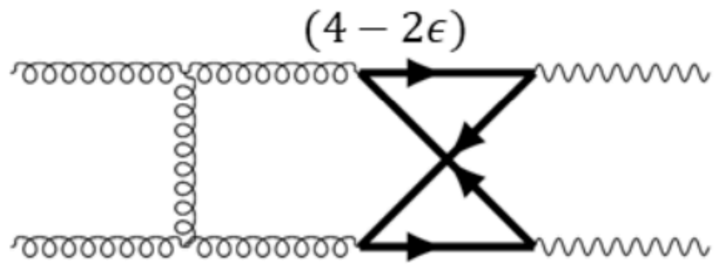
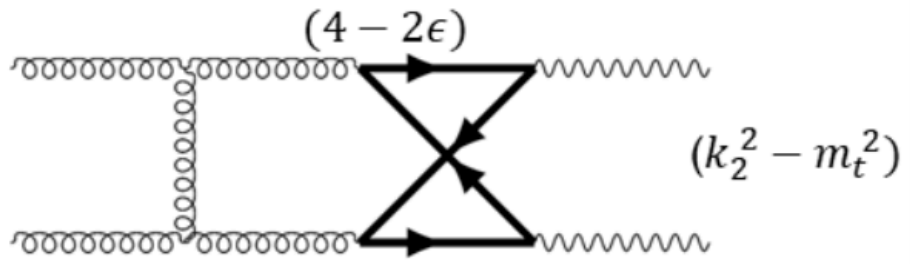
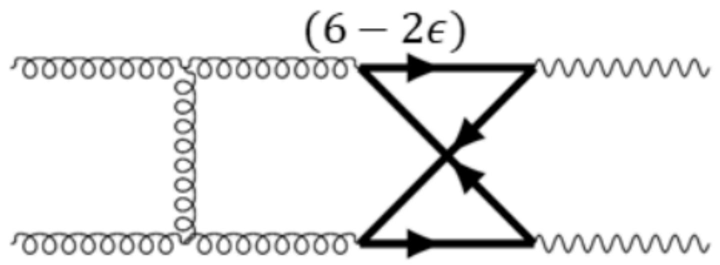
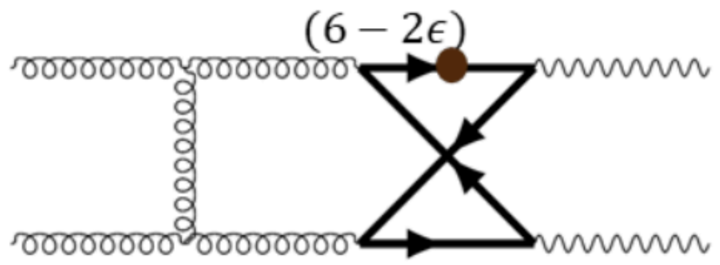
$$
\begin{aligned}
&= \frac{1}{\epsilon^8}\left(-\frac{1}{144}\right) + \frac{1}{\epsilon^7}\left(-\frac{1}{12}\right) + \frac{1}{\epsilon^6}\left(\frac{1}{24}\zeta_2 - \frac{7}{36}\right) + \frac{1}{\epsilon^5}\left(\frac{29}{24}\zeta_3 + \frac{1}{2}\zeta_2 - \frac{1}{72}\right) \\
&+ \frac{1}{\epsilon^4}\left(\frac{71}{16}\zeta_2^2 + \frac{29}{2}\zeta_3 + \frac{39}{16}\zeta_2 + \frac{335}{144}\right) + \frac{1}{\epsilon^3}\left(\frac{1819}{24}\zeta_5 - \frac{23}{6}\zeta_2\zeta_3 + \frac{213}{4}\zeta_2^2 + \frac{1211}{24}\zeta_3 + \frac{431}{48}\zeta_2 \right.\\
&\left.+ \frac{47}{18}\right) + \frac{1}{\epsilon^2}\left(-\frac{1285}{24}\zeta_3^2 + \frac{80579}{1008}\zeta_2^3 + \frac{1819}{2}\zeta_5 - 46\zeta_2\zeta_3 + \frac{25787}{160}\zeta_2^2 + \frac{417}{8}\zeta_3 - \frac{1175}{48}\zeta_2 - \frac{7277}{72}\right) \\
&+ \frac{1}{\epsilon}\left(\frac{434203}{192}\zeta_7 - \frac{7139}{24}\zeta_2\zeta_5 - \frac{54139}{120}\zeta_2^2\zeta_3 - \frac{1285}{2}\zeta_3^2 + \frac{80579}{84}\zeta_2^3 + \frac{5571}{2}\zeta_5 - \frac{9005}{24}\zeta_2\zeta_3 + \frac{967}{480}\zeta_2^2 \right.\\
&\left.- \frac{4045}{8}\zeta_3 - \frac{733}{24}\zeta_2 + \frac{57635}{72}\right) - \frac{2023}{12}\zeta_{5,3} - \frac{30581}{4}\zeta_3\zeta_5 - \frac{6829}{24}\zeta_2\zeta_3^2 + \frac{45893321}{100800}\zeta_2^4 + \frac{434203}{16}\zeta_7 \\
&- \frac{7139}{2}\zeta_2\zeta_5 - \frac{54139}{10}\zeta_2^2\zeta_3 - \frac{10706}{3}\zeta_3^2 + \frac{7987951}{3360}\zeta_2^3 + \frac{1309}{12}\zeta_5 - \frac{30317}{24}\zeta_2\zeta_3 - \frac{43847}{96}\zeta_2^2 + \frac{32335}{24}\zeta_3 \\
&+ \frac{2553}{4}\zeta_2 - \frac{334727}{72} + \mathcal{O}(\epsilon)\,.
\end{aligned}
$$

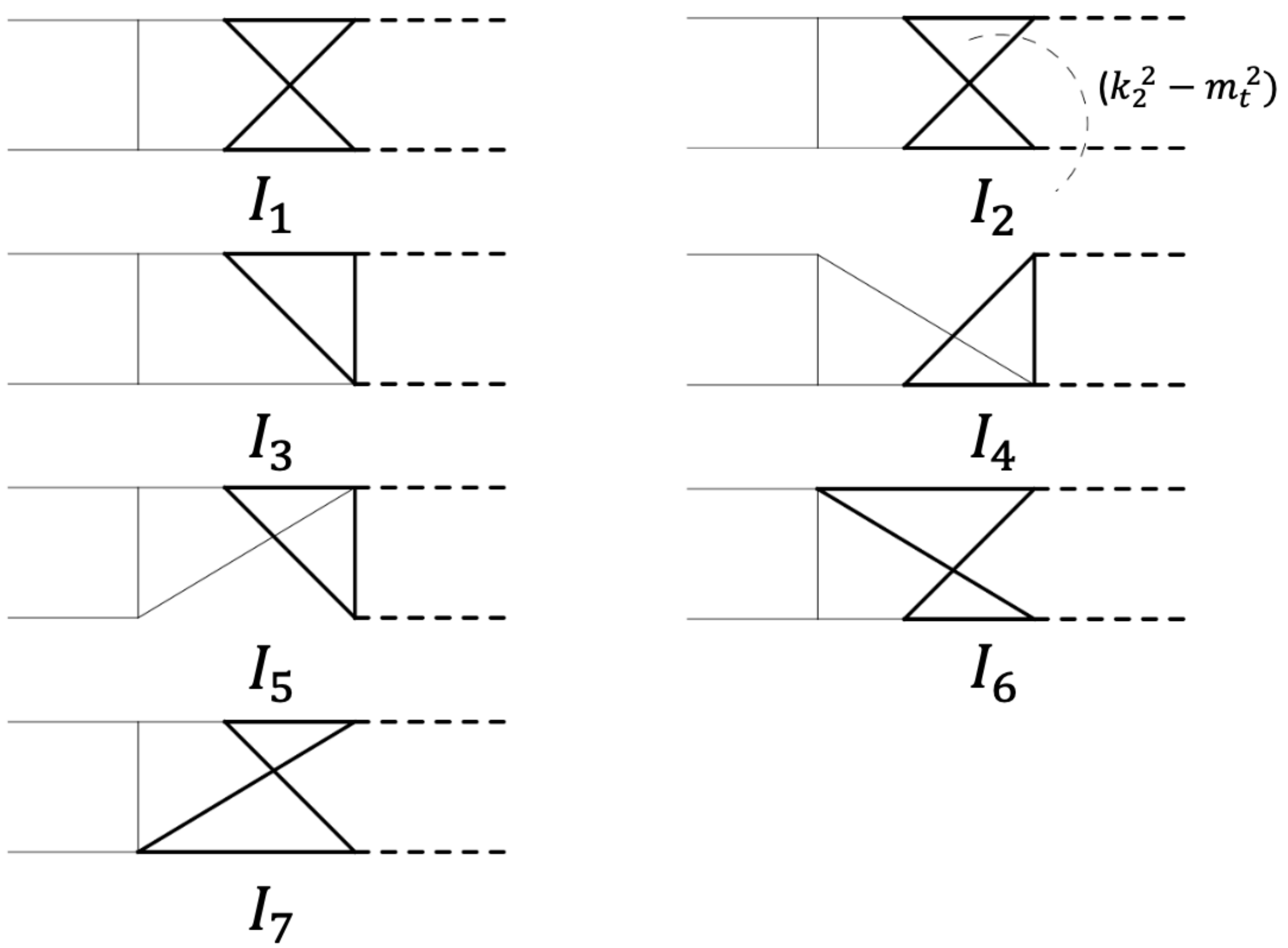
- Only 1 term for weight 6 for a nice finite integral:



$$= -\frac{3}{2}\zeta_3^2 - \frac{4}{3}\zeta_2^3 + 10\zeta_5 + 2\zeta_2\zeta_3 - \frac{1}{5}\zeta_2^2 - 6\zeta_3 + \mathcal{O}(\epsilon)$$

GENERALIZED FINITE INTEGRALS

Integral	Rel.Err.	Timing(s)
	~2*10^-3	45
	~4*10^-2	63
	~8*10^-6	55
	~8*10^-4	60
Linear combination	~1*10^-4	18



$$I = (m_z^2 - s - t)(sI_1 - I_6) + s(I_2 + I_3 - I_4 - I_5) - (m_z^2 - t)I_7$$

$$I(\nu_1,\dots,\nu_N) = (-1)^{r+\Delta t}\,\Gamma(\nu - L\,d/2) \int \left(\prod_{j\in\mathcal{N}_T} \mathrm{d}x_j\right) \left(\prod_{j\in\mathcal{N}_t} \frac{x^{\nu_j-1}}{\Gamma(\nu_j)}\right) \delta\left(1 - \sum_{j\in\mathcal{N}_T} x_j\right) \left[\left(\prod_{j\in\mathcal{N}_{\setminus T}} \frac{\partial^{|\nu_j|}}{\partial x_j^{|\nu_j|}}\right)\left(\prod_{j\in\mathcal{N}_{\Delta t}} \frac{\partial^{|\nu_j|+1}}{\partial x_j^{|\nu_j|+1}}\right) \frac{\mathcal{U}^{\nu-(L+1)d/2}}{\mathcal{F}^{\nu-L\,d/2}}\right]_{x_j=0\ \forall\ j\in\mathcal{N}_{\setminus T}} \quad (\nu_j \in \mathbb{Z}).$$

[Agarwal, AvM, Jones 2020]

Numerical integration uses pySecDec *[Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke 2017]*

See also momentum space construction in: *[Gambuti, Kosower, Novichkov, Tancredi 2023]*

- Univariate partial fraction decomposition separates singularities:

$$\frac{x}{(x-1)(x+1)^2} = -\frac{1}{4(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{4(x-1)}$$

- Iterated partial fractioning introduces spurious poles in multivariate case:

$$\frac{1}{(x-f(y))(x-g(y))} = \frac{1}{(f(y)-g(y))}\frac{1}{(x-f(y))} - \frac{1}{(f(y)-g(y))}\frac{1}{(x-g(y))},$$

for example:

$$\frac{1}{(x+y)(x-y)} = \frac{1}{2y}\frac{1}{(x-y)} - \frac{1}{2y}\frac{1}{(x+y)}.$$

- Our approach to be discussed in the following: MultivariateApart [*Heller et al ’21*]
- Related work: *[Pak ’11, Abreu et al ’19, Boehm et al ’20, Bendle et al ’21]*
- Motivation: (non-planar) amplitudes sometimes reduced by factor $O(100)$ in size with respect to common denominator representation

PARTIAL FRACTIONS VIA POLYNOMIAL REDUCTIONS

- Algorithm: write inverse denominators as $q_i = 1/d_i$, reduce polynomial in $q_1, \ldots, x_1, \ldots$ w.r.t. ideal

$$I = \langle q_1 d_1(x_1, \ldots) - 1, \ldots, q_m d_m(x_1, \ldots) - 1 \rangle$$

- Here, **polynomial reduction** means

$$p' = p - u \cdot g$$

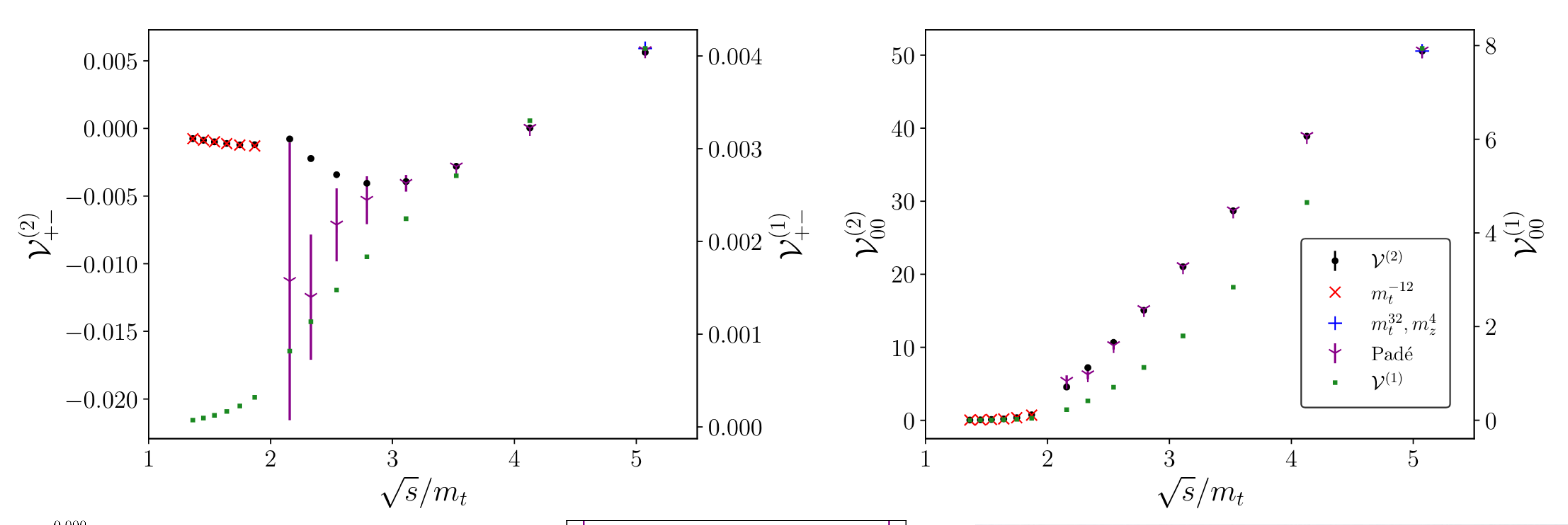
such that p' "smaller" than p for some monomial ordering, u is an arbitrary polynomial and $g \in I$

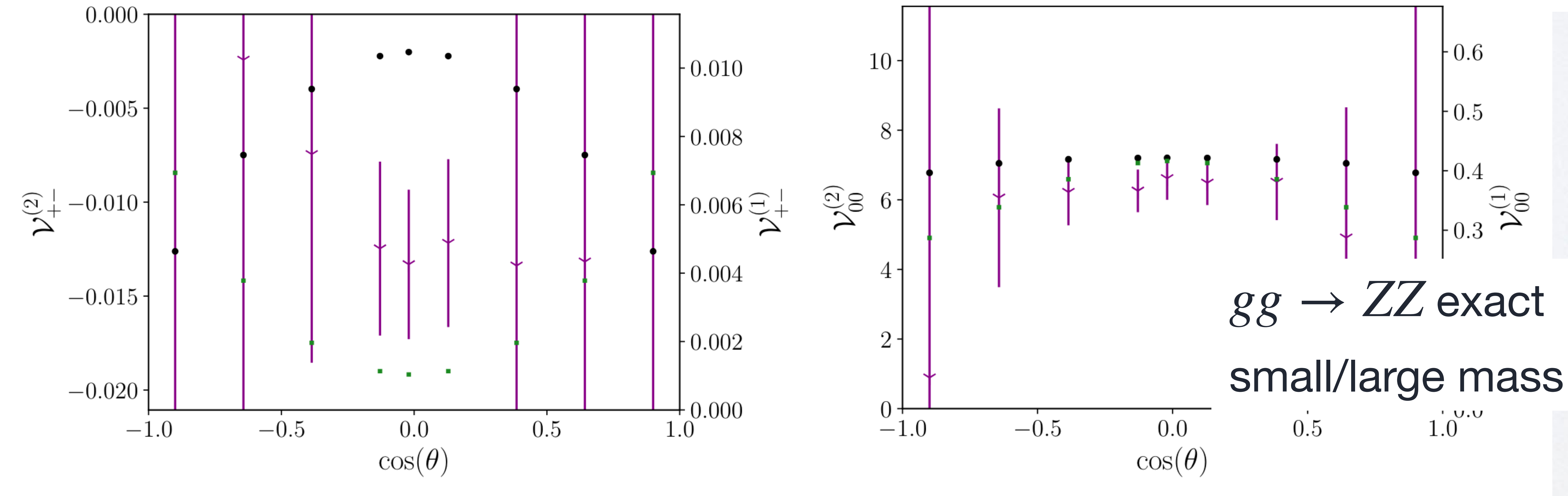
- Depending on monomial ordering we can ensure specific features of output form:
    - **Theorem I:** Result is always unique if we consider all g from a Gröbner basis
    - **Theorem II:** Sorting $q_1, \ldots$ before $x_1, \ldots$ guarantees that denominators have common zeros (Leinartas (i), *useful since it separates singular behavior)*
    - **Theorem III:** A lexicographic ordering of the $q_1, \ldots$ and $x_1, \ldots$ (separately) guarantees also algebraic independence of different denominators (Leinartas (ii), *a possible choice, but not necessarily needed)*
- Implemented in public Mathematica package: MultivariateApart [*Heller, AvM '21]*

APPLICATION OF MULTIVARIATEAPART METHOD

- Decompose: $r(x,y) = \dfrac{x-2y}{(x-y)y(x+y)}$
- Ideal: $I = \langle q_1(x-y)-1,\ \ q_2 y - 1,\ \ q_3(x+y)-1 \rangle$
- Monomal ordering: $\{\{q_3, q_1\}, \{q_2\}, \{x, y\}\}$
- Gröbner basis: $\{-1 + q_2 y, -1 + q_1 x - q_1 y, -1 + q_3 x + q_3 y, -q_1 q_2 + 2 q_1 q_3 + q_2 q_3\}$
- Reducing polynomial $r = (x-2y)q_1 q_2 q_3$ gives
$$r = -\frac{1}{2}q_1 q_2 + \frac{3}{2} q_2 q_3 = -\frac{1}{2(x-y)y} + \frac{3}{2y(x+y)}$$
- $gg \to ZZ$ top contributions:
    - 9 d-dependent, 48 kinematic denominators, up to 6th order
    - generate C++ code interfaced to pySecDec

EXACT VS EXPANSION

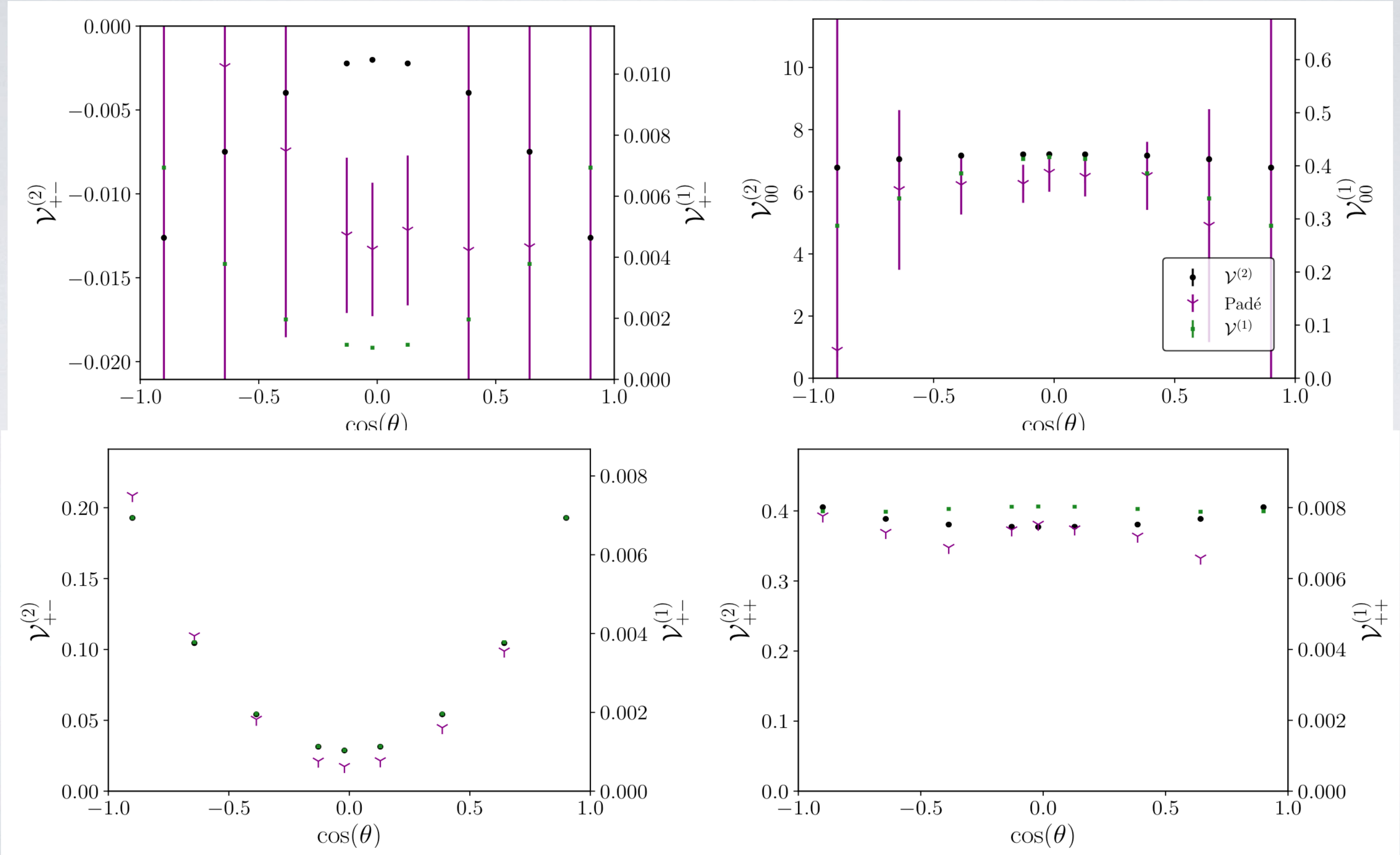




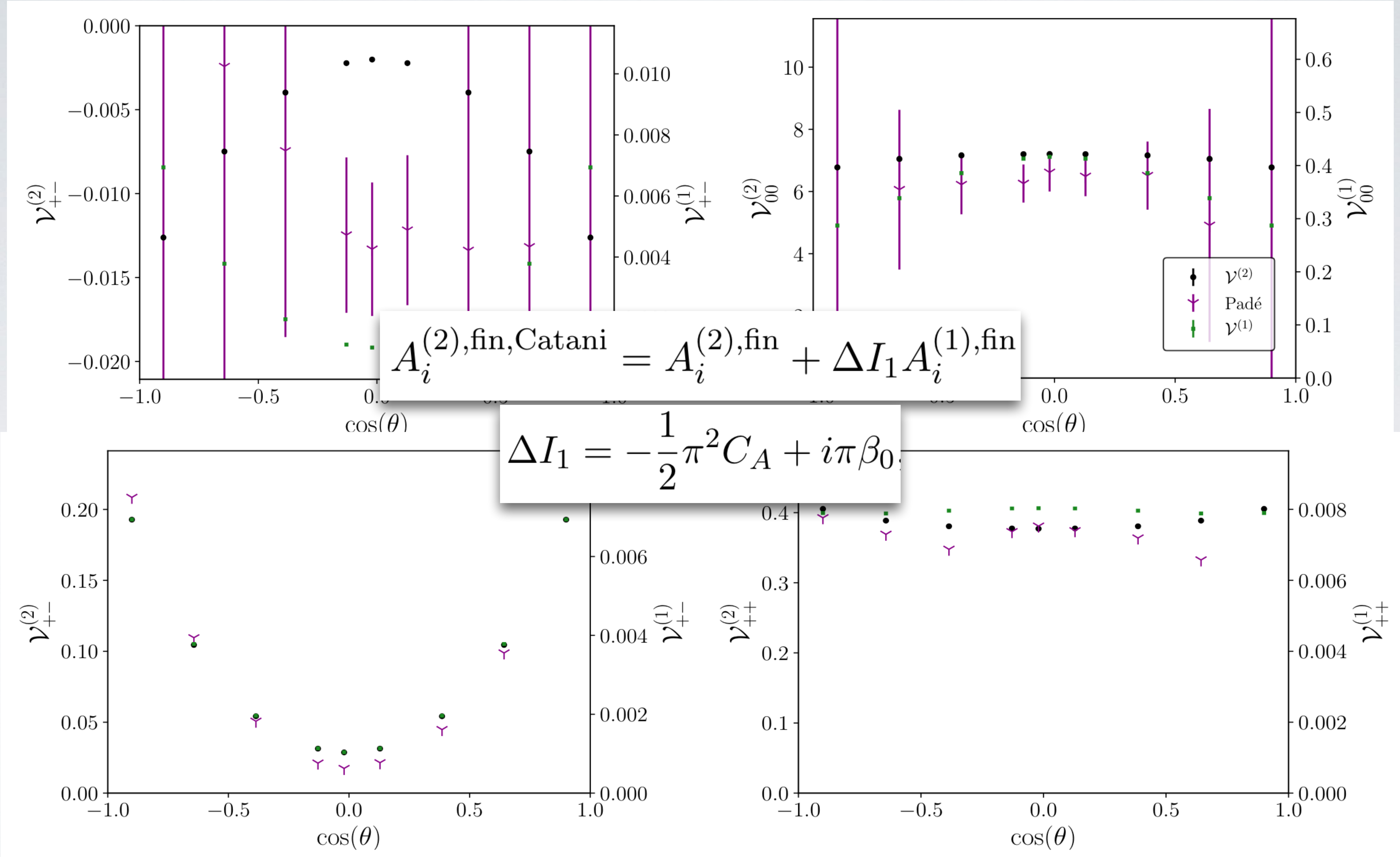
$gg \to ZZ$ exact *[Agarwal, Jones, AvM '20]*

small/large mass Padé *[Davies, Mishima, Steinhauser, Wellmann '20]*

COMPARE UPPER VS LOWER ROW



COMPARE UPPER VS LOWER ROW



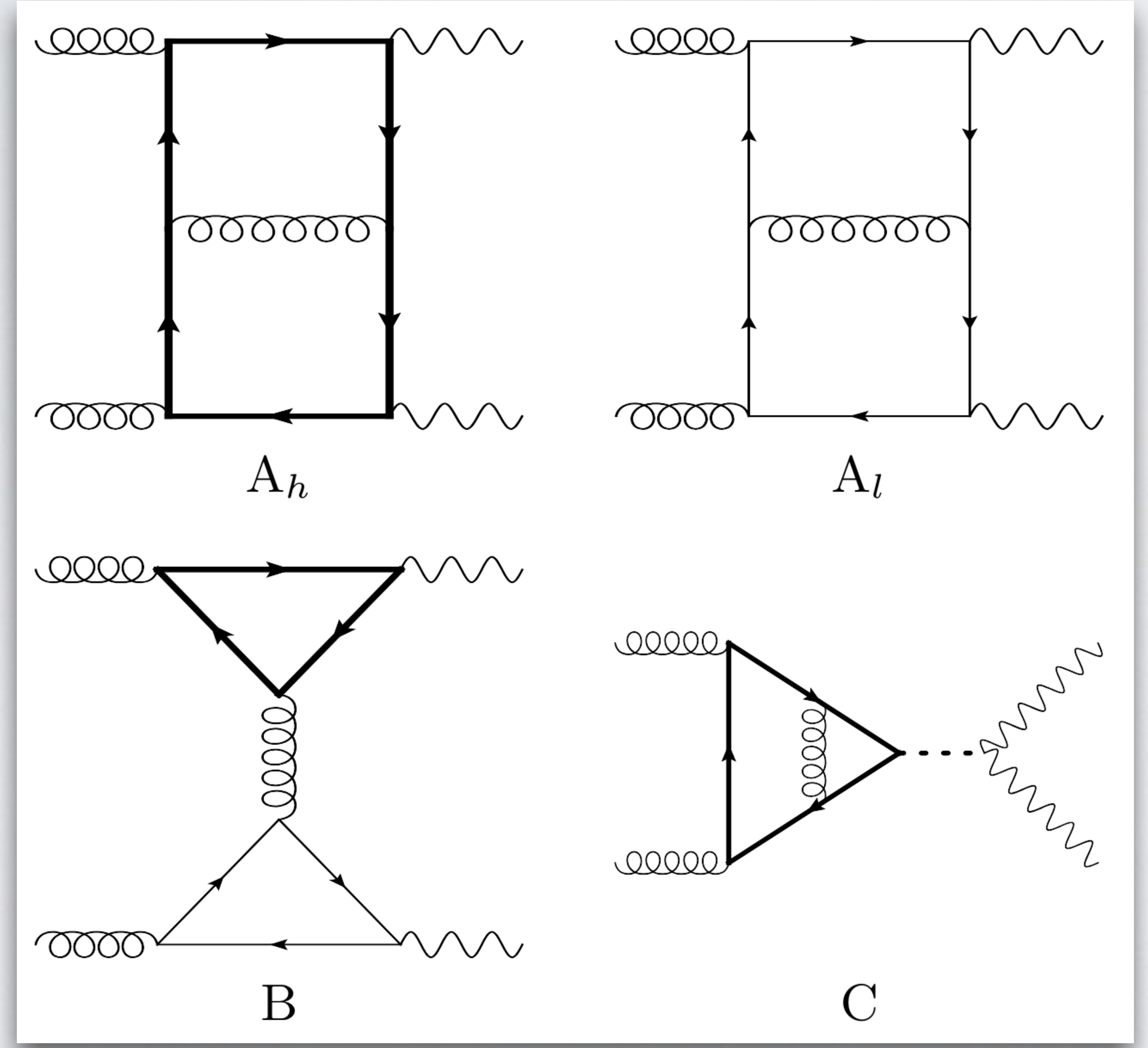
$$A_i^{(2),\mathrm{fin,Catani}} = A_i^{(2),\mathrm{fin}} + \Delta I_1 A_i^{(1),\mathrm{fin}}$$

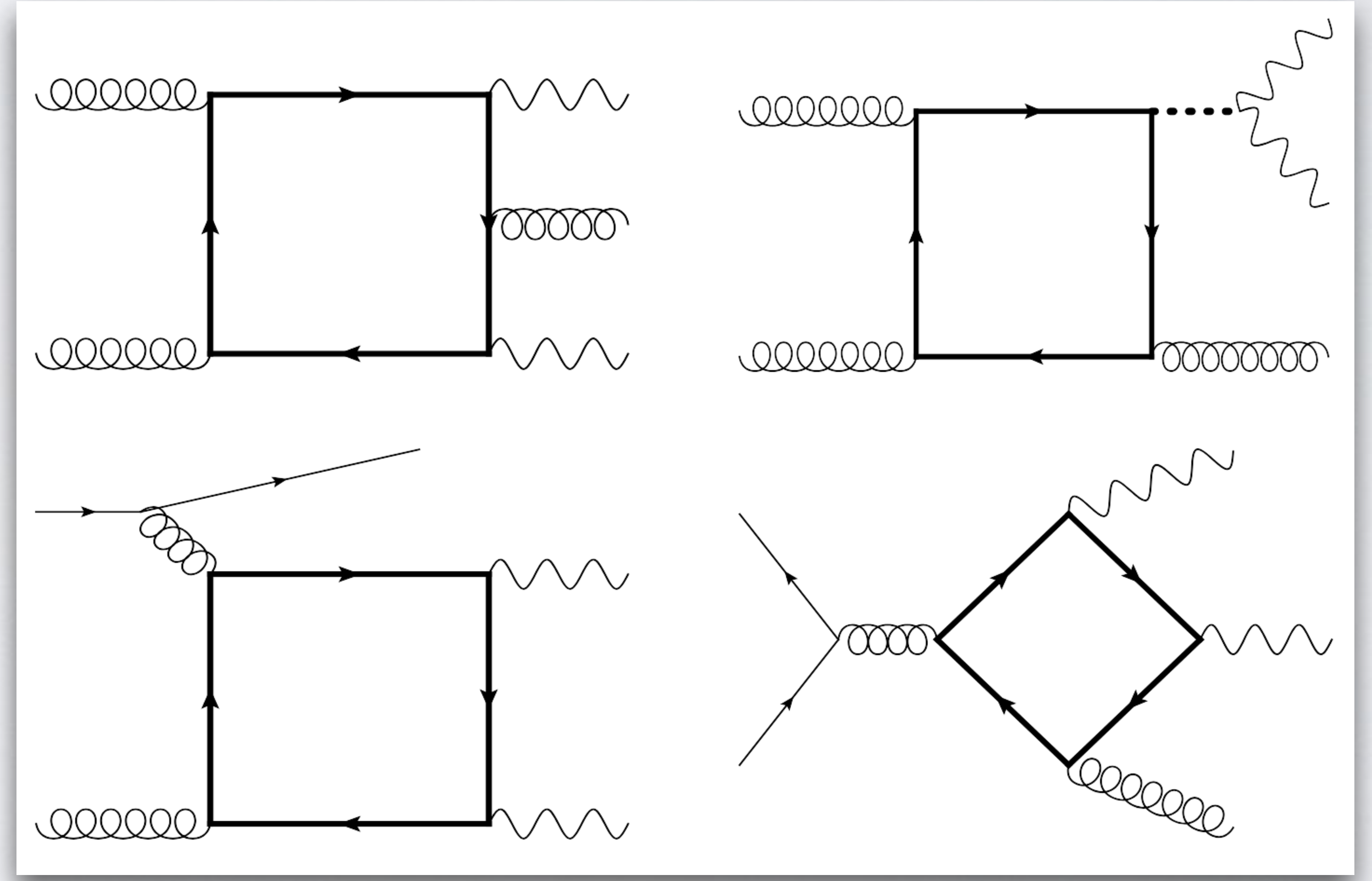
$$\Delta I_1 = -\frac{1}{2}\pi^2 C_A + i\pi\beta_0$$

ASSEMBLY OF AMPLITUDES

[Agarwal, Jones, AvM ’20,
Bronnum-Hansen, Wang ’21]

[AvM, Tancredi ’15 (VVamp),
Caola, Henn, Melnikov, Smirnov, Smirnov ’15]



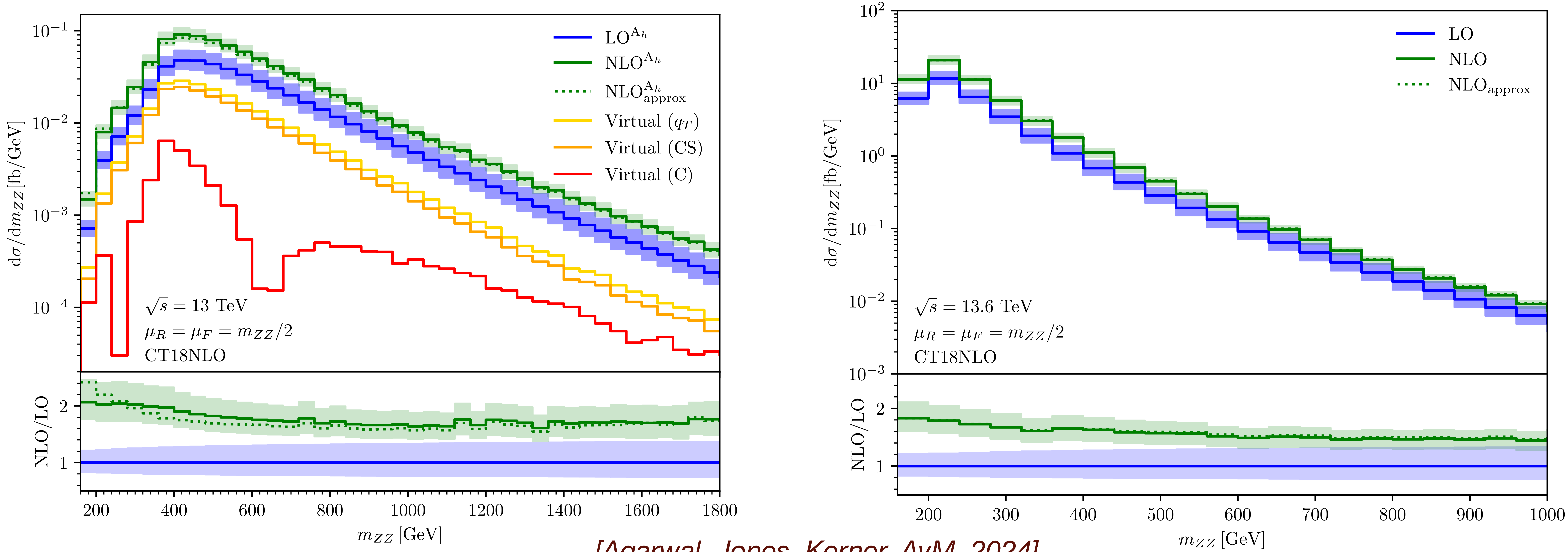


[GoSam]

[Campbell, Ellist, Czakon, Kirchner ’16;
Agarwal, Jones, AvM ’20]

[Anastasiou, Beerli, Bucherer, Daleo, Kunszt ’06;
Agarwal, Jones, Kerner, AvM ’24]

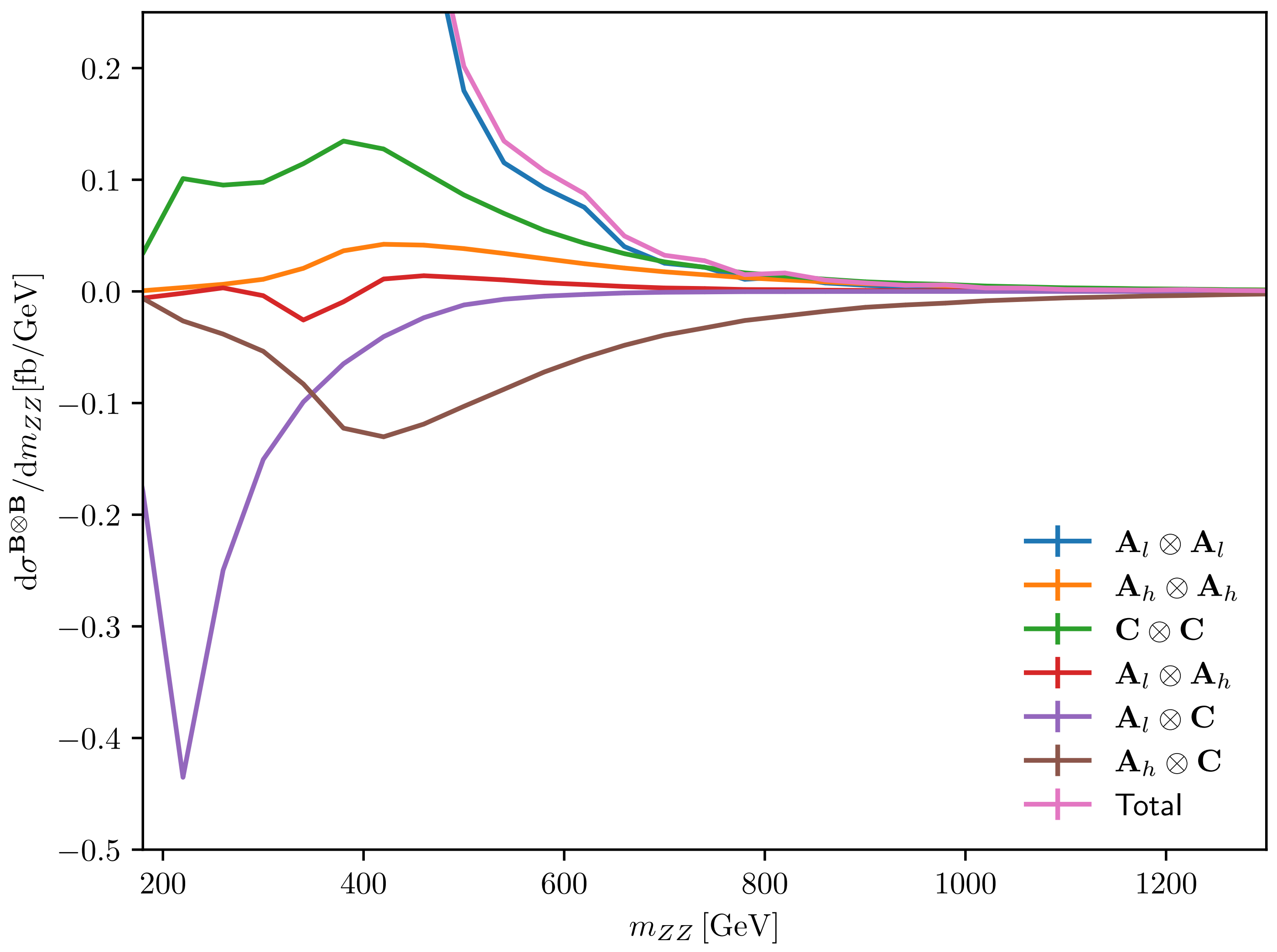
$gg \to ZZ$ INV MASS, SIZE OF VIRTUALS

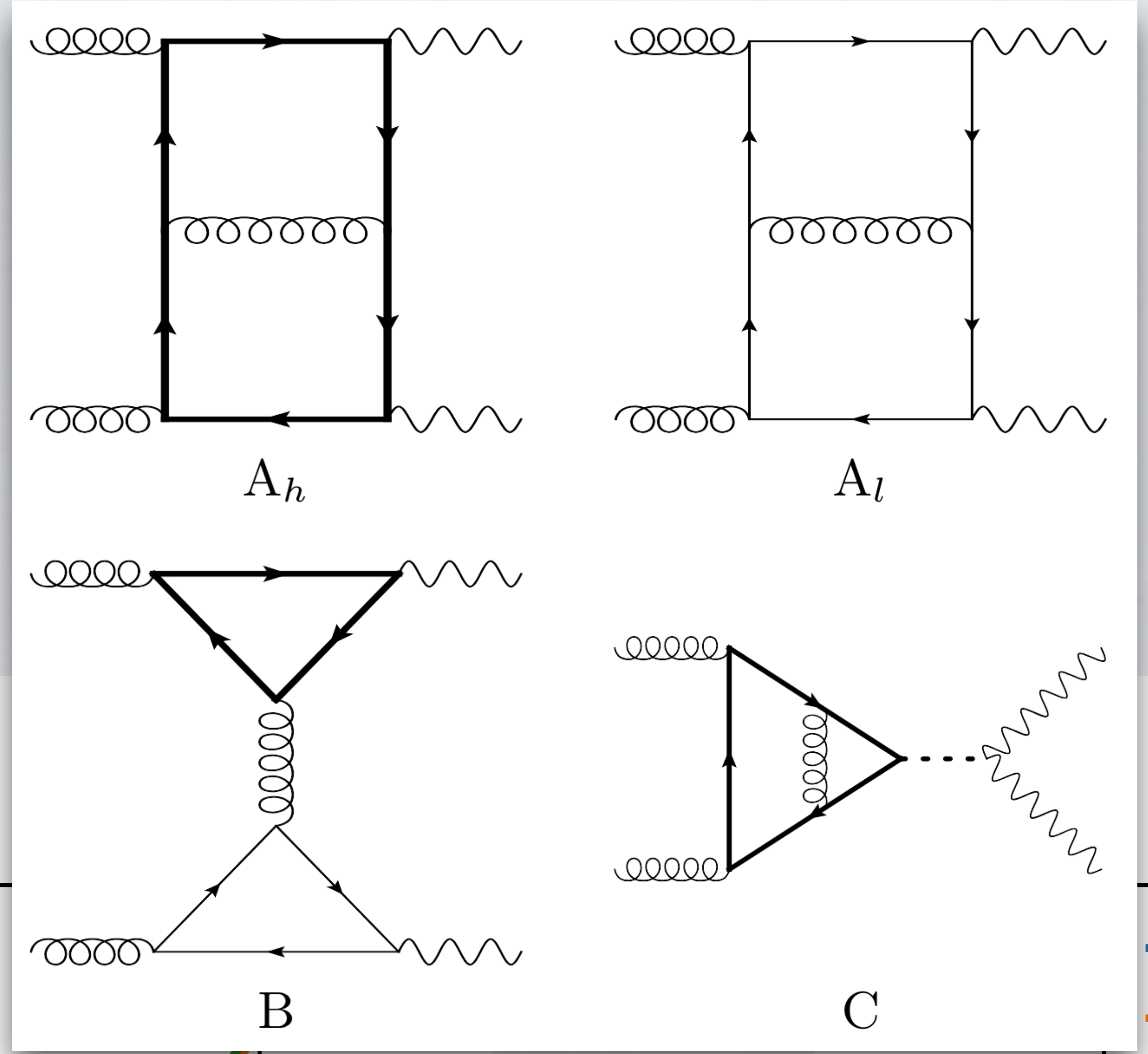


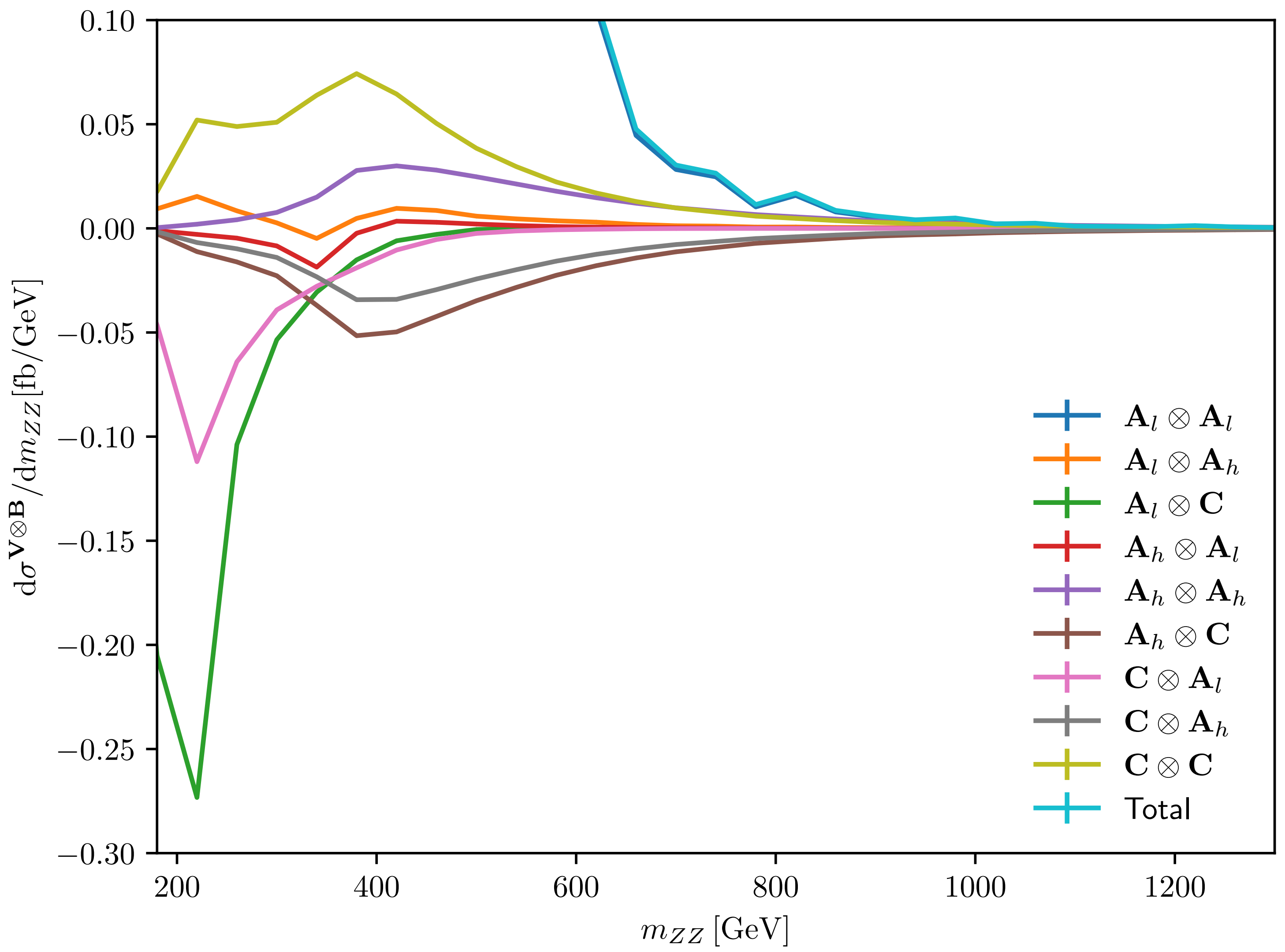
[Agarwal, Jones, Kerner, AvM, 2024]

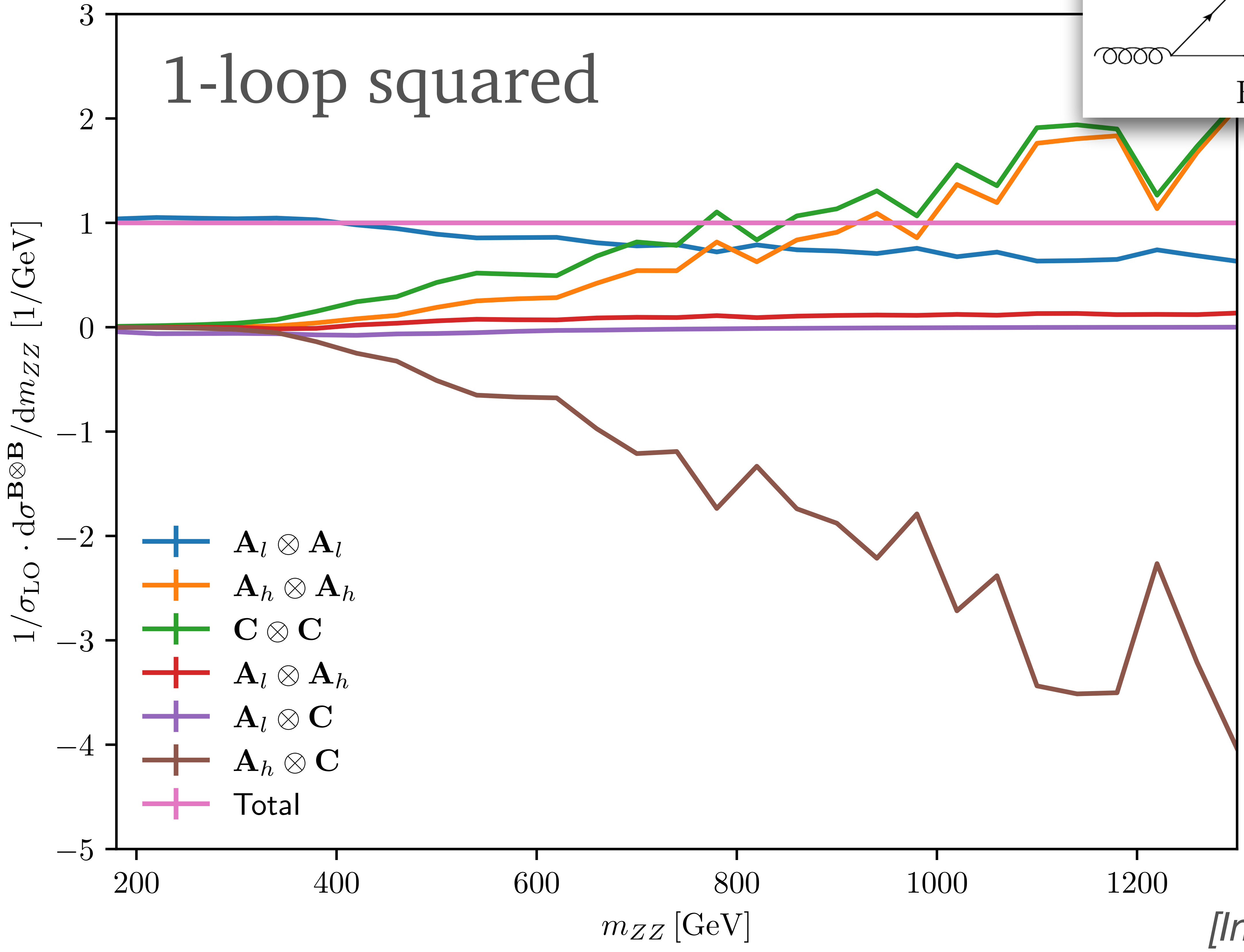
- 2-loop remainders particular small in Catani's scheme (C)
- Approximation of 2-loop massive boxes w/ massless "K-factor" works well

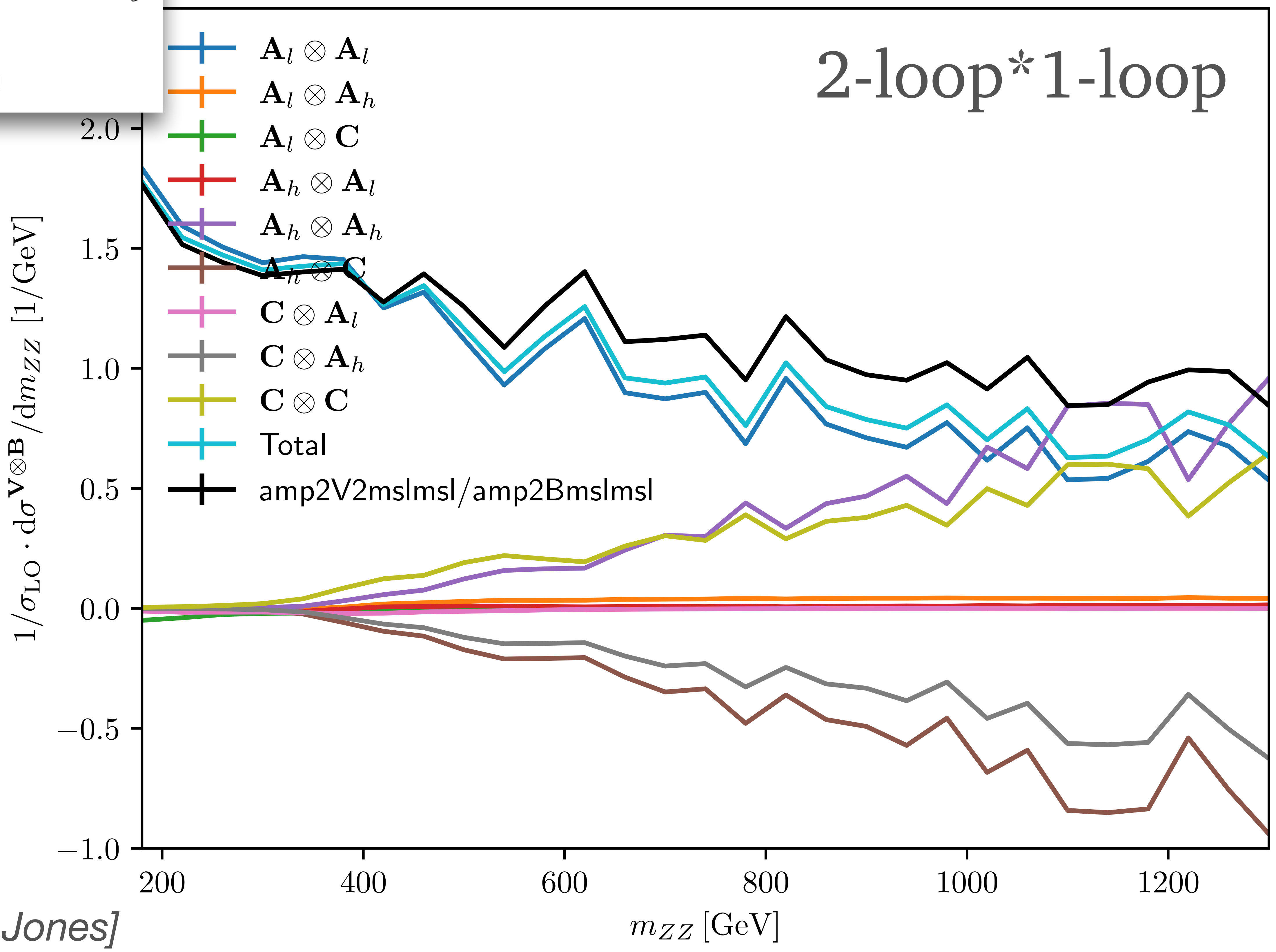
INTERFERENCES











[Image credit: Stephen Jones]

TOWARDS ALL-N, FOUR-LOOP DGLAP EVOLUTION

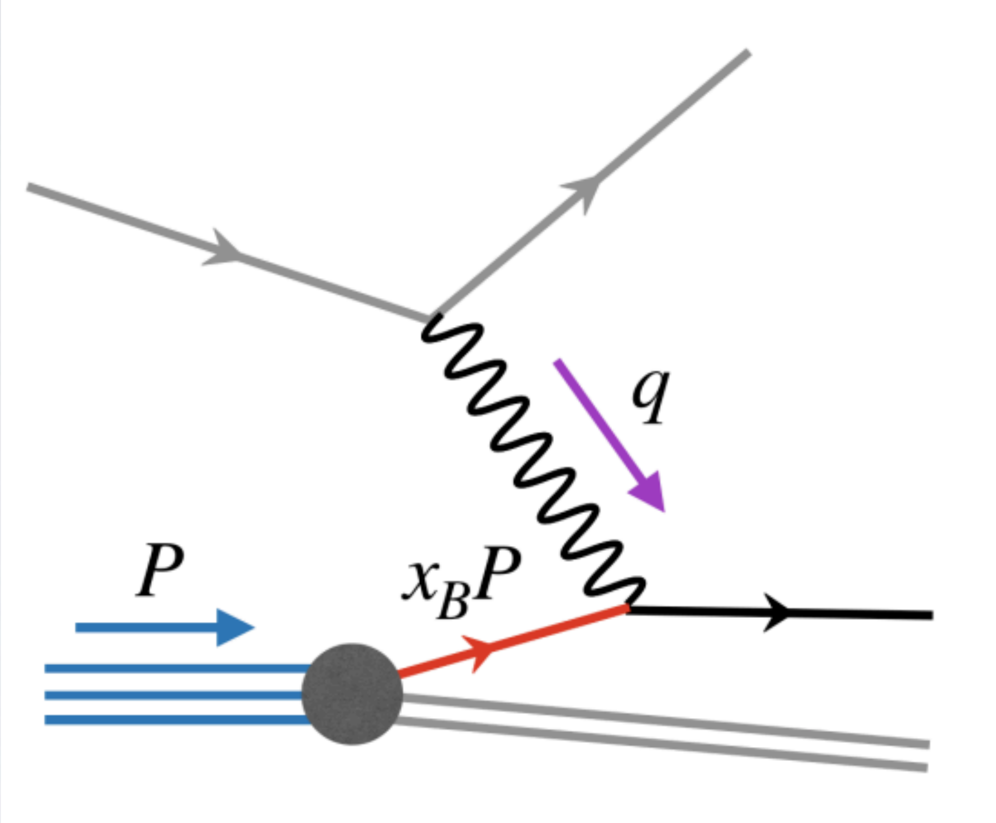
SPLITTING FUNCTIONS

- Factorization of hadronic cross section:

$$\sigma \sim \sum_k f_{k|N}(x) \otimes \sigma_k(x) \ \text{ with } x = -\frac{q^2}{2P \cdot q}$$

- Splitting functions P_{ik} govern DGLAP evaluations of PDFs:

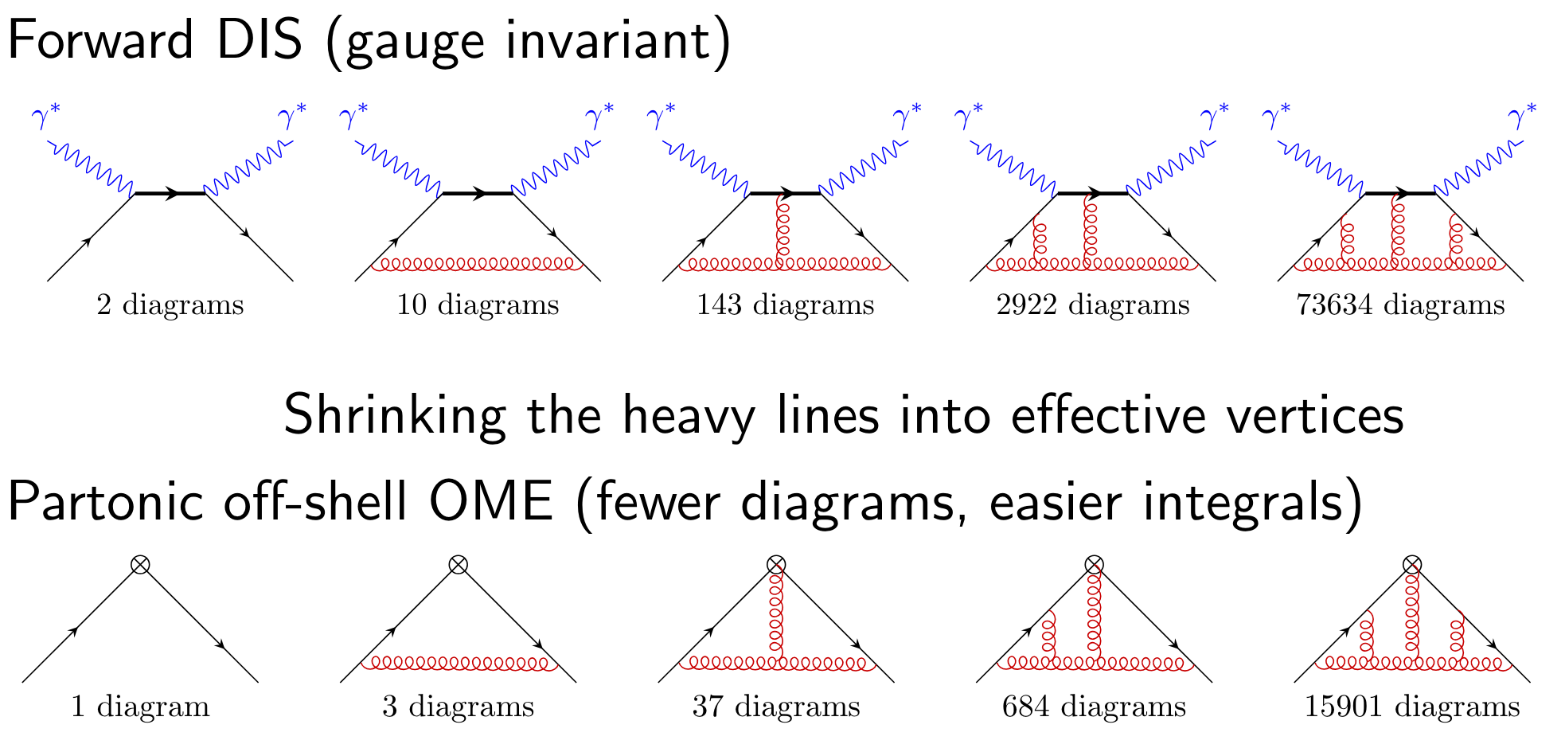
$$\frac{df_{i|N}}{d \ln \mu} = 2 \sum_k P_{ik} \otimes f_{k|N}$$



[Image credit: Tong-Zhi Yang]

- Consistent N3LO cross section requires 4-loop splitting functions, only partially known:
  - Large n_f limit *[Gracey '94,'96; Davies, Vogt, Ruijl, Ueda, Vermageren '16]*
  - Non-singlet $n \leq 16$ from off-shell OMEs *[Moch, Ruijl, Ueda, Vermaseren, Vogt '17]*
  - Singlet $n \leq 8$ from DIS *[Moch, Ruijl, Ueda, Vermaseren, Vogt '21]*
  - All channels $n \leq 10$, pure singlet $n \leq 12$, gq/qg $n \leq 20$ from off-shell OMEs *[Falcioni, Herzog, Moch, Vogt '23, '23, '24]*
  - Approximate N3LO PDF fits: see talk by Tommaso Giani
  - This talk: all-n results for pure-singlet n_f^2 and non-singlet $n_f C_F^3$ splitting functions at 4 loops

DIS VS OFF-SHELL OME



[Image credit: Tong-Zhi Yang]

SPLITTING FUNCTIONS FROM OPERATORS

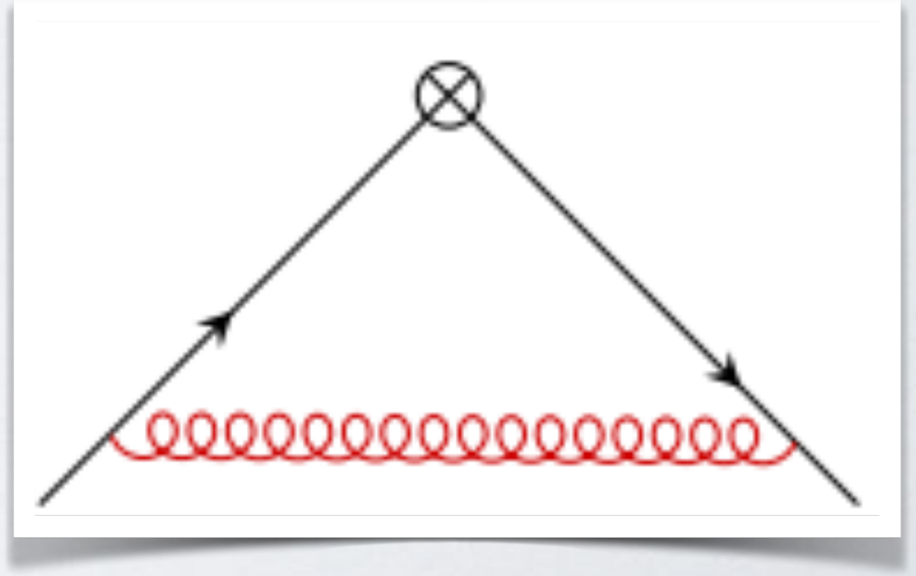
- With Mellin transform $f_q(n) = -\int_0^1 dx\, x^{n-1} f_q(x)$, $\gamma_{ij}(n) = -\int_0^1 dx\, x^{n-1} P_{ij}(x)$

  DGLAP becomes $\dfrac{df_i(n,\mu)}{d\ln\mu} = -2\sum_j \gamma_{ij}(n) f_j(n,\mu)$

- The $\gamma_{ij}(n)$ appear as **anomalous dimensions** of **twist-two operators**,

  e.g. flavor non-singlet: $O_{q,k} = \dfrac{i^{n-1}}{2}\left[\overline{\psi}\Delta_\mu \gamma^\mu (\Delta\cdot D)^{n-1}\dfrac{\lambda_k}{2}\psi\right]$

  with multiplicative renormalization $O^R_{q,k} = Z^{ns} O^B_{q,k}$ where $\dfrac{dZ^{ns}}{d\ln\mu} = -2\gamma^{ns} Z^{ns}$



- Poles of (off-shell) operator matrix elements: **efficient** way to find $f_q(n)$

SINGLET CASE AND OPERATOR MIXING

- Singlet twist-two operators:

$$O_q = \frac{i^{n-1}}{2}\left[\bar{\psi}\Delta_\mu \gamma^\mu (\Delta\cdot D)^{n-1}\psi\right]$$

$$O_g = -\frac{i^{n-2}}{2}\left[\Delta_\mu G^{a\,\mu}{}_{\nu}(\Delta\cdot D)^{n-2}_{ab}\Delta_\kappa G_b{}^{\kappa\nu}\right]$$

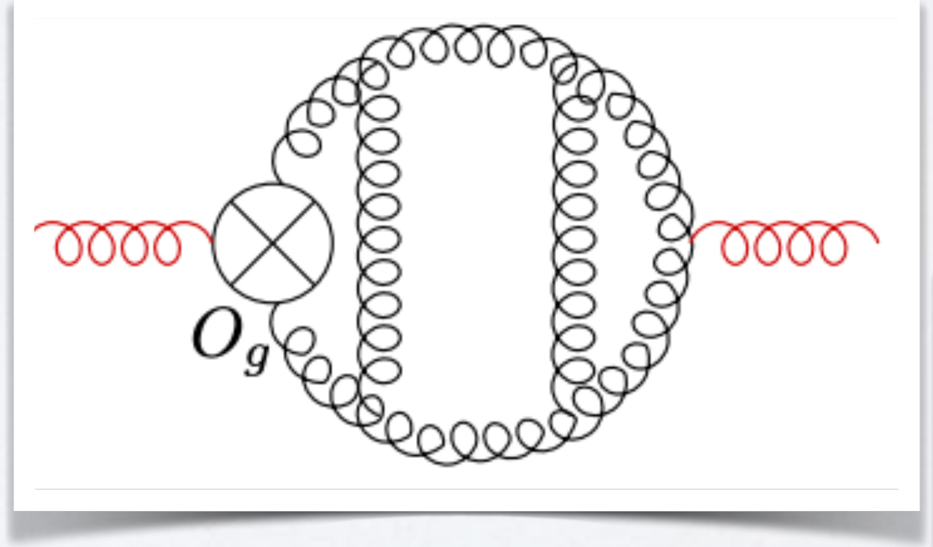
- Singlet operators mix under renormalization
- For off-shell OME, also new, unknown gauge-variant operators contribute
- Gauge-variant operators caused confusion in early literature
- Construction of operators for fixed Mellin moment n from generalized BRST: *[Falcioni, Herzog '22]*
- Our goal: **all-n results**
- Our method: directly compute **counter term Feynman rules** from multi-leg off-shell OMEs *[Gehrmann, AvM, Yang '23]*

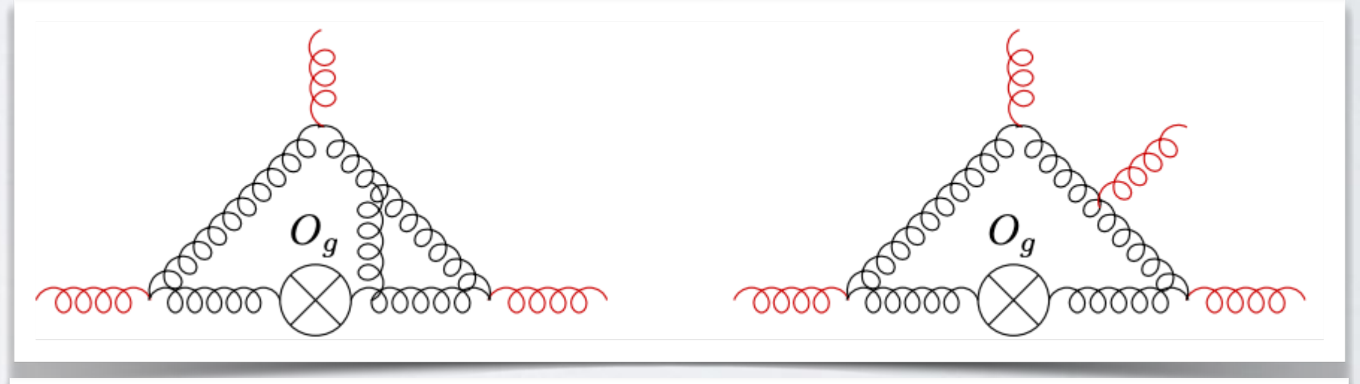
COUNTER TERMS FROM MULTI-LEG OMES

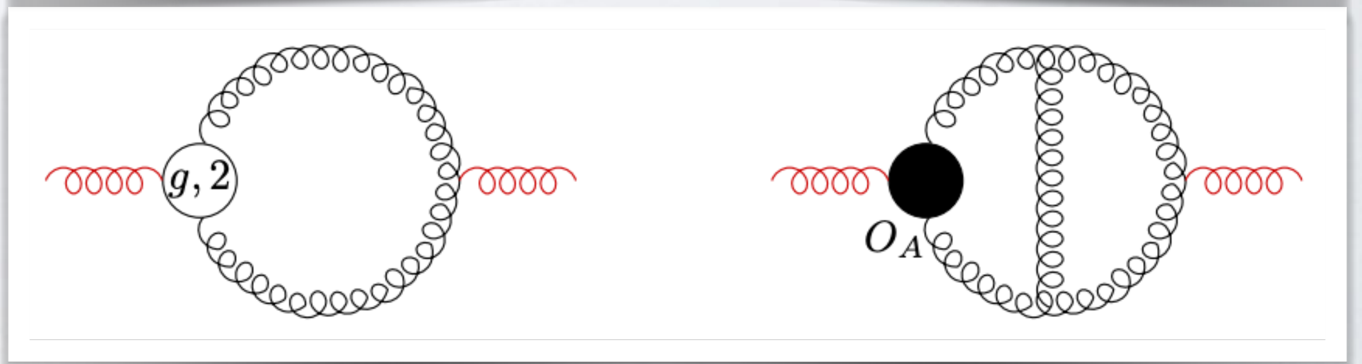
- Renormalization: $\begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^R = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ Z_{Aq} & Z_{Ag} & Z_{AA} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^B + \begin{pmatrix} [ZO]_q^{GV} \\ [ZO]_g^{GV} \\ [ZO]_A^{GV} \end{pmatrix}^B$

- Take OMEs according to $\langle j | O | j + mg \rangle$ with $j = q, g, c$ and m additional gluons
- Expand $[ZO]^{GV} = \sum_l [ZO]^{GV,(l)} \alpha_s^l$, determine counter terms from OMEs with extra legs, e.g.:

Loops \ Legs	2	3	4	5
0		$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_q, O_g
1	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g	
2	O_{ABC}	O_g		
3	O_q, O_g			







[Gehrmann, AvM, Yang ‘23]

SPLITTING FUNCTIONS @ 3-LOOPS

- Operator insertions introduce n dependent powers of scalar products
- Use **tracing parameter** t to map to standard linear propagators *[Ablinger, Blümlein, Hasselhuhn, Schneider, Wissbrock '12]*

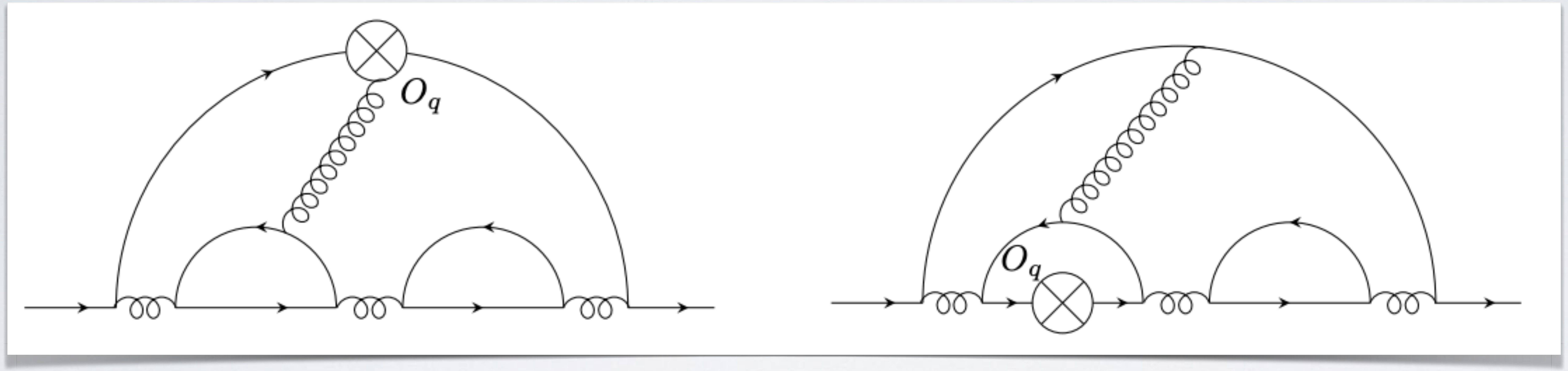
$$(\Delta \cdot p)^{n-1} \to \sum_{n=1}^{\infty} t^n (\Delta \cdot p)^{n-1} = \frac{t}{1 - t\,\Delta \cdot p}$$

allows to use standard IBP technology

- We applied our method to **3-loop splitting functions**, computation in general R_ξ gauge
- Differential equations in t, find ϵ factorized form using Canonica and Libra, boundary val. known
- Complicated counter terms, involve generalized harmonic sums
- Gauge parameter ξ drops out, full agreement with *[Moch, Vermaseren, Vogt '04, '04]*

PURE SINGLET @ 4-LOOPS: N_f^2, ALL-N

- Four-loop contributions for quark, with two or three closed fermion loops
  [Gehrmann, AvM, Sotnikov, Yang ’23]

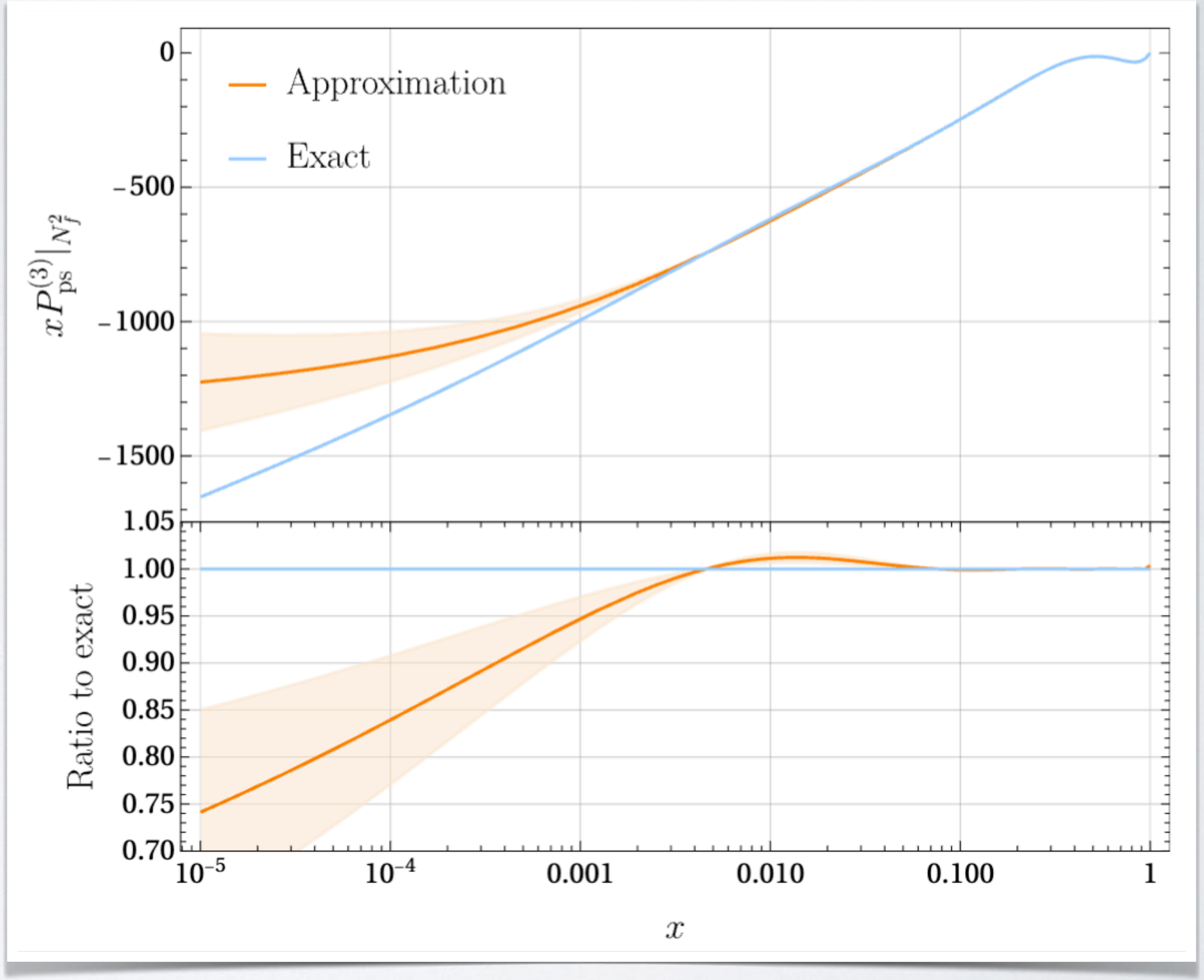


  (singlet and non-singlet, also non-planar)

- Use syzygies, compute with linear algebra
- Finred with finite field sampling to derive differential equations, reduction of amplitude
- Simple analytical result for splitting functions in terms of HPLs and powers of x

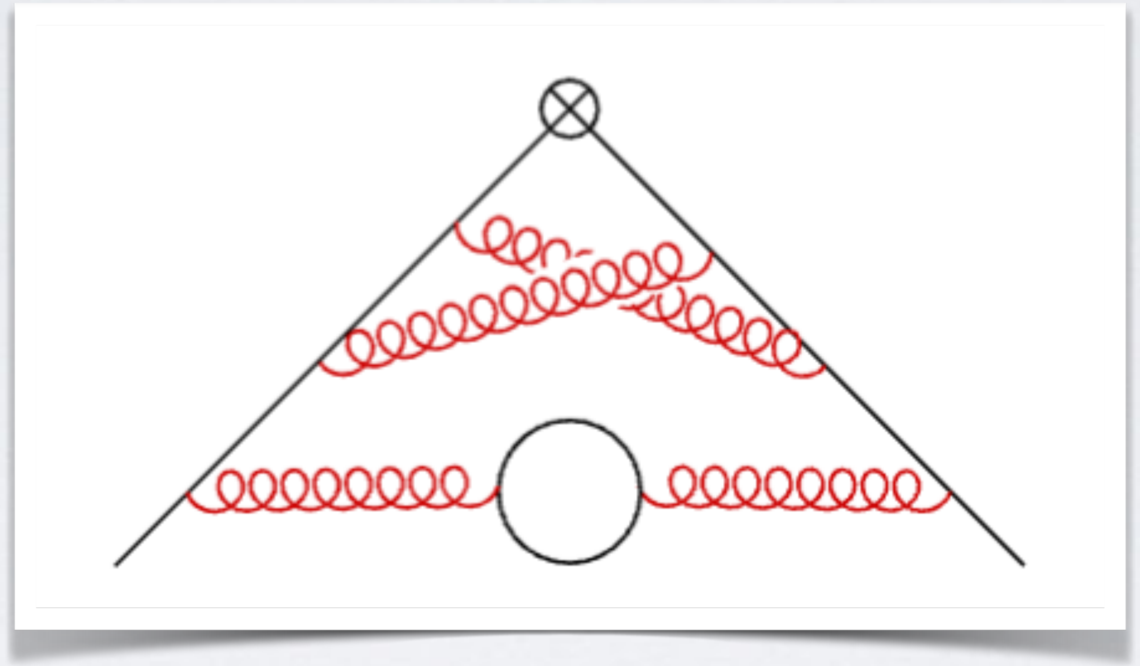
PURE SINGLET @ 4-LOOPS: N_f^2, ALL-N

- $n \leq 20$ by *[Falcioni, Herzog, Moch, Vogt '23]*
- partial information for $x \to 0$: *[Catani, Hautmann '94; Davies, Kom, Moch, Vogt '22]*
- leading terms for $x \to 1$: *[Soar, Moch, Vermaseren, Vogt '09]*
- Generate fit similar to *[Falcioni, Herzog, Moch, Vogt '23]*, compare to all-n result
- All-n result improves small x knowledge



[Gehrmann, AvM, Sotnikov, Yang '23]

NON-SINGLET @ 4-LOOPS: $n_f C_F^3$



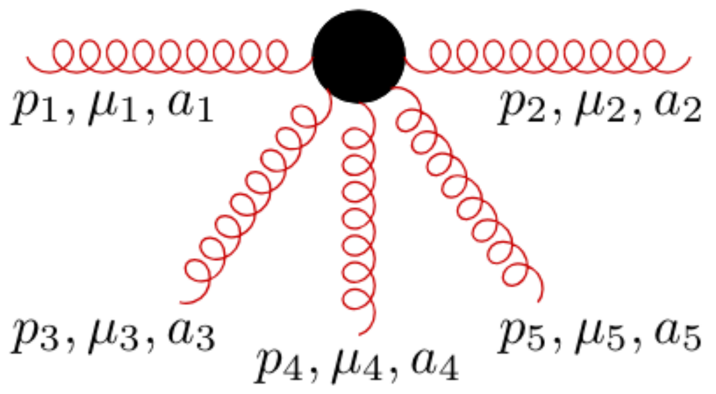
- Calculated $n_f C_F^3$ non-singlet, four-loop splitting functions with exact x dependence

  [Gehrmann, AvM, Sotnikov, Yang '23]

- fixed moments $n \leq 16$ *[Moch, Ruijl, Ueda, Vermaseren, Vogt '17]*

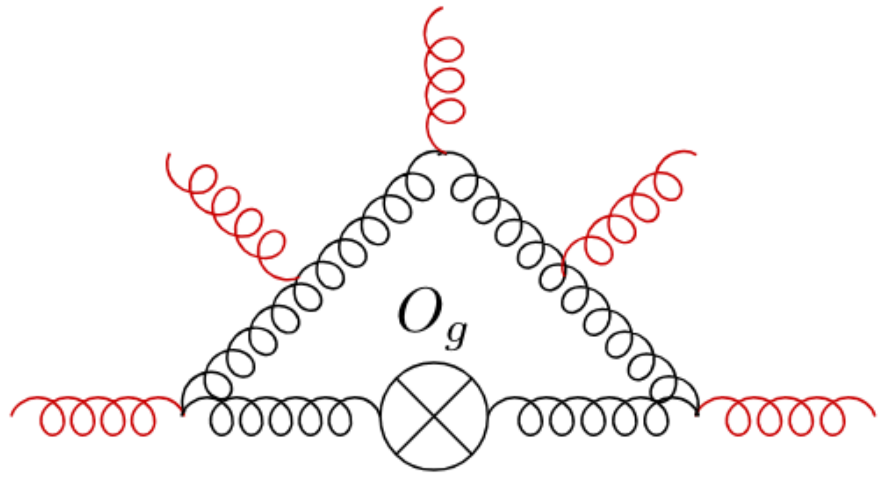
- ζ_4, ζ_5 terms conjectured *[Davies, Vogt '17]*

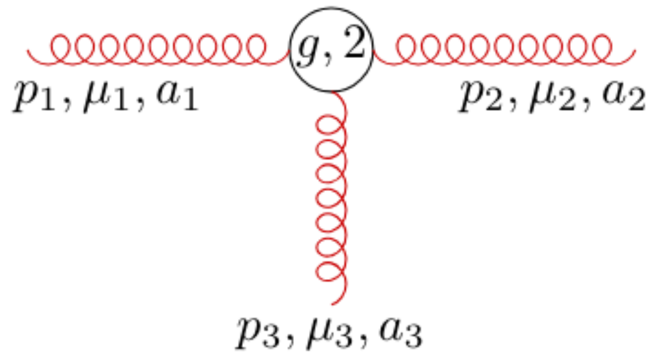
REMAINING SINGLET CT @ 4 LOOPS



$$\to \frac{1+(-1)^n}{2} i\, g_s^3 \Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4} p_1^{\mu_5} \Bigg[\frac{1}{C_A} f^{aa_1a_2} d_4^{aa_3a_4a_5} \Bigg\{ \frac{3}{32} \sum_{j_1=0}^{n-4} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} (-\Delta\cdot p_1)^{n-4-j_1} (-\Delta\cdot(p_1+p_2))^{j_1-j_2} \times (\Delta\cdot(p_4+p_5))^{j_2-j_3} (\Delta\cdot p_5)^{j_3} + \cdots \Bigg\} + 11 \text{ color structures} \Bigg] + 30 \text{ Lorentz Structures} \quad \text{17074 lines}$$

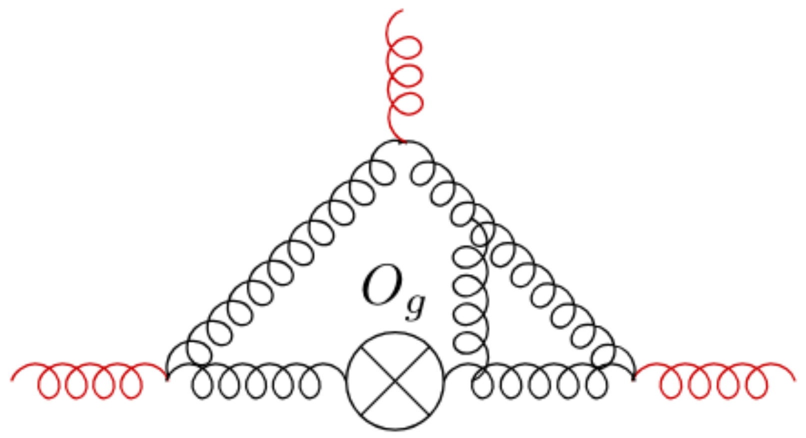
extracted from





$$\to 2 i g_s C_A^2 f^{a_1a_2a_3} \frac{1+(-1)^n}{256 n(n-1)} \frac{(\Delta\cdot p_1)^{n-2}}{\Delta\cdot p_2} \Big(\Delta^{\mu_2}\Delta^{\mu_3} p_1^{\mu_1} \Delta\cdot p_1 + \cdots \Big) \left\{ \frac{F_{-2,0}(\xi, z_1, n)}{\epsilon^2} + \frac{F_{-1,0}(\xi, z_1, n)}{\epsilon} \right\}, \quad z_1 = \frac{\Delta\cdot p_2}{\Delta\cdot p_1}$$

extracted from



- Feynman rules: general. harmonic sums
- IBPs with polylogs ?
- Additional tracing parameter

NON-SINGLET @ 4-LOOPS: $n_f C_F^3$

- limit $x \to 0$: $\ln^4(x), \ln^5(x)$ for $P_{ns}^{(3),+}(x)$ *[Davies, Kom, Moch, Vogt '22]*
- limit $x \to 1$: $P_{ns}^{(3),+} \approx P_{ns}^{(3),-} \approx A_4 \left[\frac{1}{1-x}\right]_+ + B_4 \delta(1-x) + C_4 \ln(1-x) + D_4 - A_4 + \mathcal{O}(1-x)$

$$A_4\Big|_{N_f C_F^3} = \frac{592}{3}\zeta_3 - 320\zeta_5 + \frac{572}{9}, \quad B_4\Big|_{N_f C_F^3} = 224\zeta_3^2 - \frac{256}{3}\zeta_2\zeta_3 - 308\zeta_3 + 162\zeta_2 - 204\zeta_4 + 912\zeta_5 - \frac{6434}{9}\zeta_6 + 32 \simeq 80.779482\,,$$

$$C_4\Big|_{N_f C_F^3} = 256\zeta_3 - \frac{880}{3}, \quad D_4\Big|_{N_f C_F^3} = 80\zeta_2 - 192\zeta_3 + \frac{464\zeta_4}{3} - \frac{638}{3}$$

  - **Cusp anomalous dimension** A_4 analytic *[Grozin '18; Henn, Korchemsky, Mistlberger, '19; AvM, Panzer, Schabinger '20]*
  - **Virtual anomalous dimension** B_4 numeric *[Das, Mach, Vogt '19]*
  - **Collinear anomalous dimension** analytic *[Agarwal, AvM, Panzer, Schabinger '21,'22]*
  - From B_4 and coll. anom. dim. find **rapidity anomalous dimension** *[Moult, Zhu, Zhu '21; Duhr, Mistlberger, Vita '21]*
  - $C = A^2$, $D = A(B + \beta/(2a_s))$ *[Dokshitzer, Marchesini, Salam '05; Basso, Korchemsky '06]* confirmed for $n_f C_F^3$

CONCLUSIONS

- **$gg \to ZZ$:**
  - NLO top-quark contributions with exact two-loop amplitudes
  - using massless K factor captures bulk (for sufficiently inclusive)
  - destructive interference Higgs / non-Higgs contributions
  - size of finite remainder strongly dependent on subtraction scheme
- **Four-loop splitting functions:**
  - efforts by several groups to compute exact x dependence
  - off-shell OME conceptually challenging, but computationally advantageous

EXTRA SLIDES

Syzygy Based IBPs Without Numerators

[Lee-Pomeransky '13] representation:

$$I(\nu_1,\ldots,\nu_N) = \mathcal{N}\left[\prod_{i=1}^{N}\int_0^\infty \mathrm{d}x_i x_i^{\nu_i-1}\right] G^{-d/2} \qquad \text{with } G = \mathcal{U}+\mathcal{F}$$

[Bitoun, Bogner, Klausen, Panzer '17]: define (twisted) Mellin Transform

$$\mathcal{M}\{f\}(\nu) := \left(\prod_{k=1}^{N}\int_0^\infty \frac{x_k^{\nu_k-1}\mathrm{d}x_k}{\Gamma(\nu_k)}\right) f(x_1,\ldots,x_N)$$

Feynman integrals are Mellin transforms:

$$\tilde{I}(\nu) = \mathcal{M}\left\{G^{-d/2}\right\}(\nu)$$

with $\nu = (\nu_1,\ldots,\nu_N)$ and $\tilde{I}(\nu) = \Gamma[(L+1)d/2-\nu]I(\nu)$ (remark: similar for Baikov's rep.)

Properties of Mellin transform

1. $\mathcal{M}\{\alpha f + \beta g\}(\nu) = \alpha\mathcal{M}\{f\}(\nu) + \beta\mathcal{M}\{g\}(\nu)$
2. $\mathcal{M}\{x_i f\}(\nu) = \nu_i \mathcal{M}\{f\}(\nu + e_i\}$
3. $\mathcal{M}\{-\partial_i f\}(\nu) = \mathcal{M}\{f\}(\nu - e_i\}$ (proof: partial integration + surface term is zero)

Define shift operators

$$(\hat{i}^+ F)(\nu_1,\ldots,\nu_N) = \nu_i F(\nu_1,\ldots,\nu_i+1,\ldots,\nu_N)$$
$$(\hat{i}^- F)(\nu_1,\ldots,\nu_N) = F(\nu_1,\ldots,\nu_i-1,\ldots,\nu_N)$$

which form Weyl algebra, $[\hat{i}^+,\hat{j}^-] = \delta_{ij}$

Shift relations from annihilators

[Lee '14; Bitoun, Bogner, Klausen, Panzer '17]: a differential operator P which annihilates $G^{-d/2}$

$$P\, G^{-d/2}$$

generates via the substitutions $x_i \to \hat{i}^+$, $\partial_i \to -\hat{i}^-$ a shift relation according to

$$\mathcal{M}\{P\, G^{-d/2}\} = 0$$

In fact, *every* shift relation is related in this way.

consider annihilators beyond linear order:

$$\left[c_0 + \sum_{i=1}^{N} c_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{N} c_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \dots \right] G^{-d/2} = 0$$

4-loop 3-point integrals w/ higher order annihilators, see *[Lee, AvM, Schabinger, Smirnov, Smirnov, Steinhauser '23]*

determine $c_0(x_1, \dots, x_N), \dots$ via syzygy equations:

$$c_0 \left[-\frac{2}{d} G^2 \right] + \sum_{i=1}^{N} c_i \left[G \frac{\partial G}{\partial x_i} \right] + \sum_{i,j=1}^{N} c_{ij} \left[G \frac{\partial^2 G}{\partial x_i \partial x_j} + \left(-\frac{d}{2} - 1 \right) \frac{\partial G}{\partial x_i} \frac{\partial G}{\partial x_j} \right] + \dots = 0$$

Syzygies generate linear relations for Feynman integrals:

$$\left(\left[c_0(\hat{1}^+, \dots, \hat{N}^+) - \sum_{i=1}^{N} c_i(\hat{1}^+, \dots, \hat{N}^+) \hat{i}^- + \sum_{i,j=1}^{N} c_{ij}(\hat{1}^+, \dots, \hat{N}^+) \hat{i}^- \hat{j}^- + \dots \right] \tilde{I} \right) (\nu_1, \dots, \nu_N) = 0$$