N-jettiness beam functions at N^{3}LO in QCD

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based on [arXiv:1809.06300] [arXiv:1904.02433] [arXiv:1910.10059] [arXiv:2211.05722]
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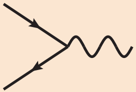
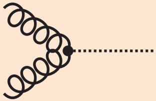
Colour singlet production at the LHC

Colour singlet production: many interesting processes

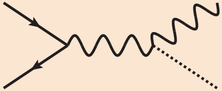
Drell-Yan/single vector boson production



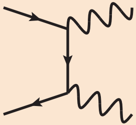
Single Higgs production



Associated Higgs production



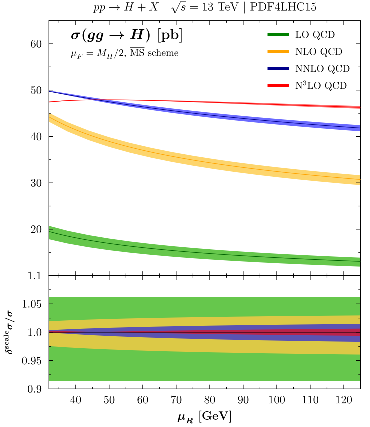
Vector boson pair production



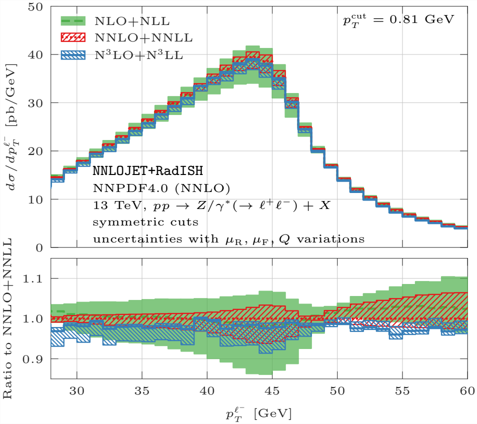
. . .

Important backgrounds to searches and interesting themselves
$\rightarrow$ PDFs, W mass, EW mixing angle, Higgs width / couplings / decays, anomalous gauge boson couplings, ...

Precision frontier



[Baglio, Duhr, Mistlberger, Szafron '22]



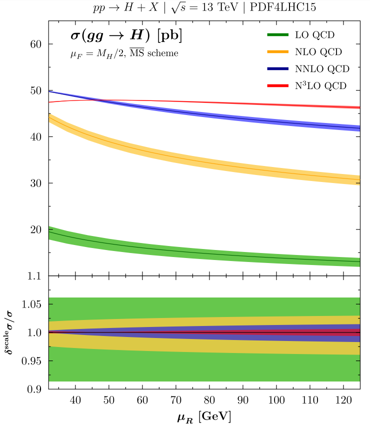
[Chen et al. '22]

Increasing experimental precision

$\to$ Interesting to push theoretical predictions

$\to$ Explore limits of perturbation theory

Precision frontier



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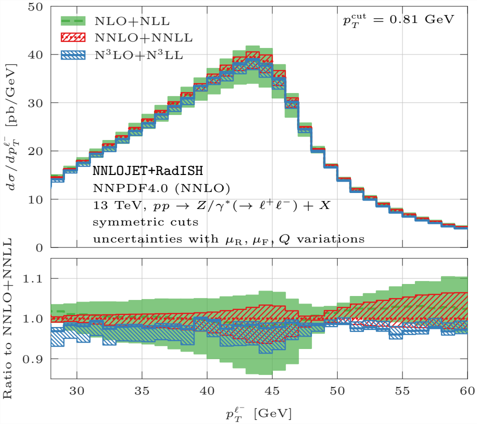


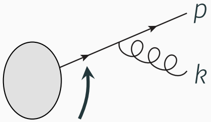
[Chen et al. '22]

Status of N^{3}LO at LHC

- Some inclusive cross-sections are known (Higgs & DY)
- First differential distributions started to appear

Infra-red singularities and slicing and subtraction schemes

IR singularities in fixed-order calculations



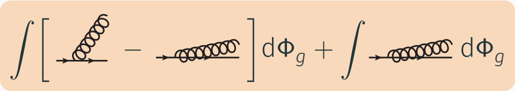
$$\frac{1}{(p-k)^2} = \frac{1}{E_p E_k (1-\cos\theta)}$$

- Appear in soft or collinear limits of massless particles
- Cancel between real and virtual contributions
- Numerical phase space integration requires scheme to treat those singularities

Subtraction schemes

$$\int_0^1 \mathrm{d}x \, \frac{f(x)}{x^{1+\varepsilon}} = \int_0^1 \mathrm{d}x \, \frac{f(x)-f(0)}{x^{1+\varepsilon}} + \int_0^1 \mathrm{d}x \, \frac{f(0)}{x^{1+\varepsilon}}$$

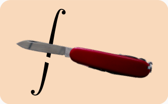
- Subtract behaviour in singular limit
- Add back integrated term



Slicing schemes

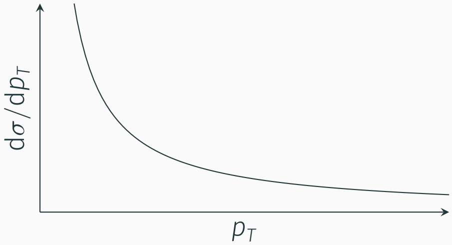
$$\int_0^1 \mathrm{d}x \, \frac{f(x)}{x^{1+\varepsilon}} = \int_\Lambda^1 \mathrm{d}x \, \frac{f(x)}{x^{1+\varepsilon}} + \int_0^\Lambda \mathrm{d}x \, \frac{f(0)}{x^{1+\varepsilon}}$$

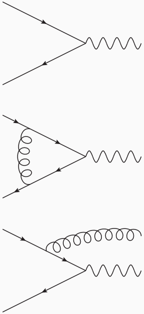
- Use observable to isolate most singular configuration
- Approximate $f(x)$ below threshold



Phase space slicing

Example: $q\bar{q} \to Z$ with p_T of the Z as slicing variable at NLO





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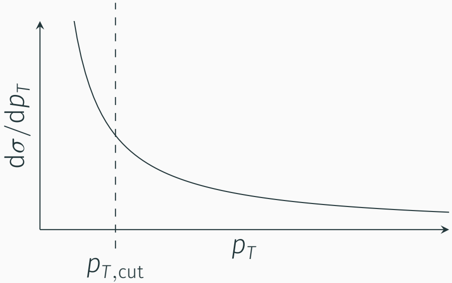

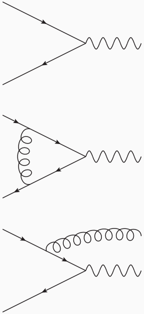

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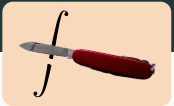

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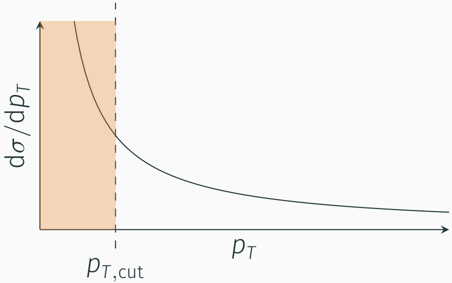
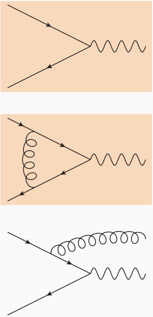
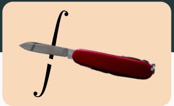




Below $p_{T,\mathrm{cut}}$:

- Born and virtual diagrams contribute
- Real diagram only contributes in soft/collinear limit

Phase space slicing



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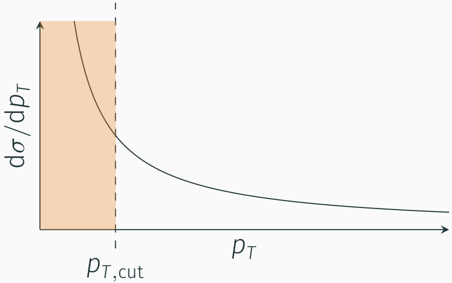
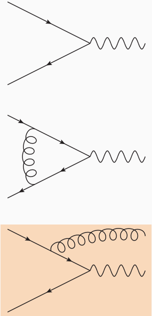




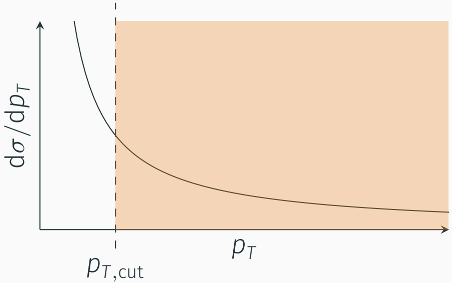
Below $p_{T,\text{cut}}$:

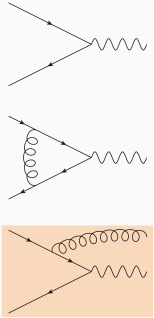
- Born and virtual diagrams contribute
- Real diagram only contributes in soft/collinear limit

$\to$ Expand cross-section in small p_T limit

Phase space slicing

Example: $q\bar{q} \rightarrow Z$ with p_T of the Z as slicing variable at NLO





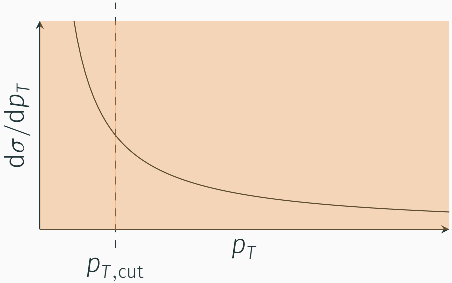
Above $p_{T,\text{cut}}$:

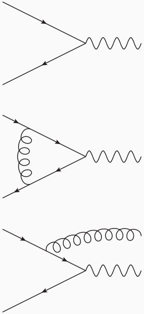
- Only real emission diagram contributes
- No soft or collinear singularities

$\rightarrow$ LO calculation for $q\bar{q} \rightarrow Z + j$ with $p_T > p_{T,\text{cut}}$

Phase space slicing

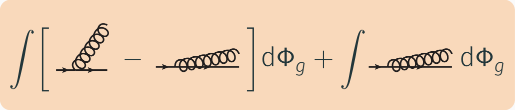
Example: $q\bar{q} \to Z$ with p_T of the Z as slicing variable at NLO





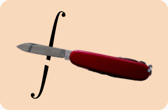
Sum of both regions is independent of $p_{T,\mathrm{cut}}$ if $p_{T,\mathrm{cut}}$ is small enough (or expansion includes sufficiently many terms)

Subtraction and slicing beyond NLO



Subtraction schemes

- Many NNLO subtraction schemes exist
- Work quite efficiently in many situations
- Not yet available at N^3LO



Slicing schemes

- Conceptually simpler
- Numerical performance can be demanding (cut-off must be very small)
- At N^3LO currently the most promising method

Phase space slicing at N^3LO

General idea

- Use slicing to isolate the fully unresolved configuration
- Above cutoff: Use NNLO calculation for $V + j$ with cut
- Below cutoff: Use approximation (factorisation theorem)

Conventional slicing observables

- Transverse momentum p_T of colour singlet state
- N-jettiness (zero-jettiness for colour singlet production)

What is N-jettiness?

Observable to describe: “How much does an event look like an N jet event?”

$\tau_N = 0$

Yes, this looks like an N jet event.

$\tau_N > 0$

There are more than N jets in this event.

What is N-jettiness?

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$\tau_N = 0$

Yes, this looks like an N jet event.

$\tau_N > 0$

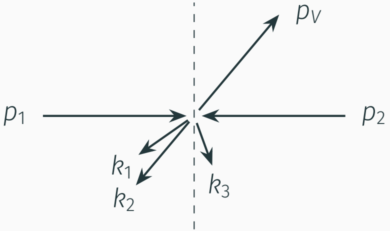
There are more than N jets in this event.

Example: Zero-jettiness (N=0)

Definition: $\tau_0 = \sum_j \min_{i \in \{1,2\}} \frac{p_i \cdot k_j}{Q_i}$

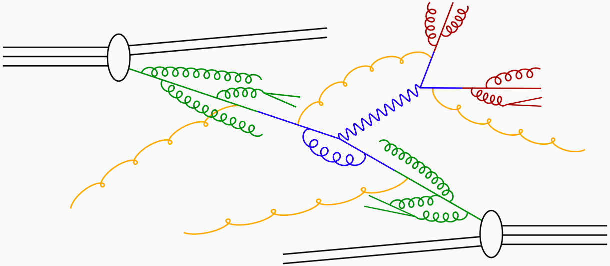
Q_i: Normalisation scales

$p_i \cdot k_j = p_i^0 k_j^0 (1 - \cos\theta_{ij})$



Zero-jettiness τ_0 vanishes if *all* real emissions become soft or collinear with initial state momenta

What are beam functions?



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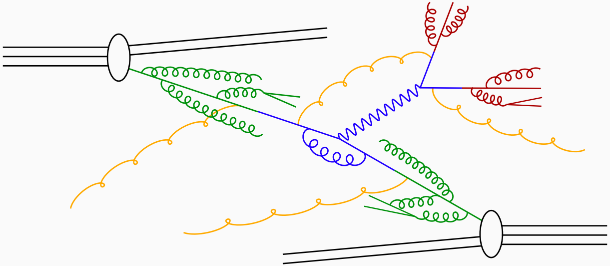
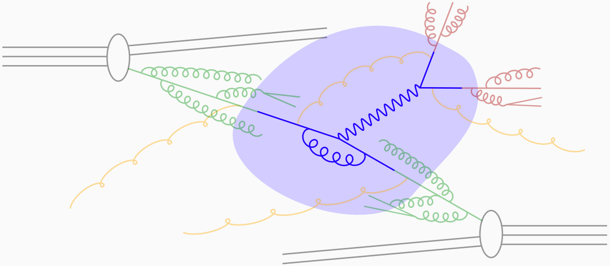


Factorisation theorem: $d\sigma \sim B \otimes B \otimes S \otimes H \otimes d\sigma_{LO} \otimes J \otimes J$

- Factorisation often occurs if kinematics become Born-like
- Enforced by limits of suitable observables
  - q_T of color-singlet system
  - N-jettiness
  - ...

What are beam functions?

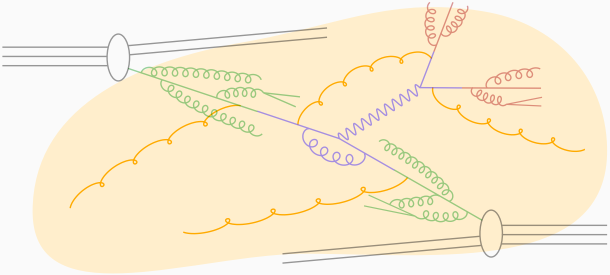


Factorisation theorem: $\mathrm{d}\sigma \sim B \otimes B \otimes S \otimes H \otimes \mathrm{d}\sigma_{\mathrm{LO}} \otimes J \otimes J$

Hard function H

- Describes hard scattering process
- Process dependent

What are beam functions?

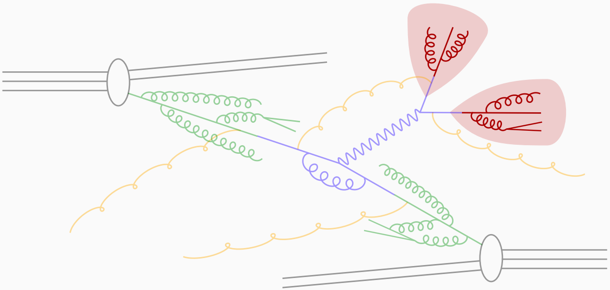


Factorisation theorem: $\mathrm{d}\sigma \sim B \otimes B \otimes S \otimes H \otimes \mathrm{d}\sigma_{\mathrm{LO}} \otimes J \otimes J$

Soft function S

- Describes soft radiation
- Process independent

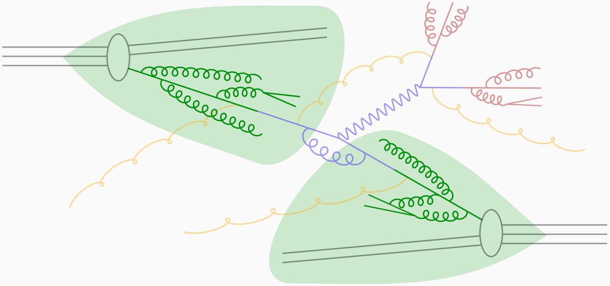
What are beam functions?



Factorisation theorem: $\mathrm{d}\sigma \sim B \otimes B \otimes S \otimes H \otimes \mathrm{d}\sigma_{\mathrm{LO}} \otimes J \otimes J$

Jet functions J

- Describe collinear radiation in the final state (jets)
- Process independent

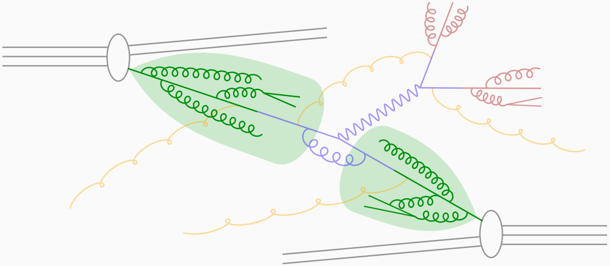


Factorisation theorem: $\mathrm{d}\sigma \sim B \otimes B \otimes S \otimes H \otimes \mathrm{d}\sigma_{\mathrm{LO}} \otimes J \otimes J$

Beam functions B

- Describe collinear radiation off the initial state
- Similar to PDFs, but more differential: $f(z, \mu)$ vs. $B(z, t, \mu)$
- Non-perturbative objects, but …

What are beam functions?



Factorisation theorem: $\mathrm{d}\sigma \sim B \otimes B \otimes S \otimes H \otimes \mathrm{d}\sigma_{\mathrm{LO}} \otimes J \otimes J$

Beam functions B

- ...can be related to PDFs via matching relation:

$$B_i(z, t, \mu) = \sum_j \mathcal{I}_{ij}(z, t, \mu) \otimes f_j(z, \mu)$$

- Matching coefficients $\mathcal{I}_{ij}(z, t, \mu)$ can be calculated perturbatively

Beam function $B(t, x, \mu)$

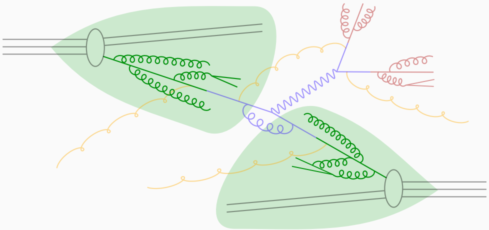
- Non-perturbative object, similar to PDFs
- Depends on
  - momentum fraction x
  - transverse virtuality $t = -(p_*^2 - k_\perp^2)$
- Can be related to PDFs via convolution

$$B_i(t, x, \mu) = \int_0^1 \mathrm{d}z \sum_{j \in \{q, \bar{q}, g\}} \underbrace{I_{ij}(t, z, \mu)}_{\text{matching coeff.}} \underbrace{f_j\left(\frac{x}{z}, \mu\right)}_{\text{PDFs}}$$

- At leading order:

$$I_{ij}^{\mathrm{LO}}(t, z, \mu) = \delta(1 - z)\delta(t)\delta_{ij}$$

$$\Rightarrow B_i^{\mathrm{LO}}(t, z, \mu) = f_i(z, \mu)\delta(t)$$



$\rightarrow$ Goal: Compute matching coefficient at N^3LO

How to calculate matching coefficients I_{ij}?

Beam function and PDFs have operator definitions in SCET:

$$B_i \sim \langle P(p)|\mathcal{O}_i(t, xp, \mu)|P(p)\rangle$$

$$f_j \sim \langle P(p)|\mathcal{Q}_j(x' p, \mu)|P(p)\rangle$$

with $|P(p)\rangle$ proton states.

Matching relation is an operator relation (OPE)

$$B_i = \sum_j I_{ij} \otimes f_j$$

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Define partonic beam functions B_{ik} and partonic PDFs f_{jk} with parton states $|p_k(p)\rangle$

How to calculate matching coefficients I_{ij}?

Beam function and PDFs have operator definitions in SCET:

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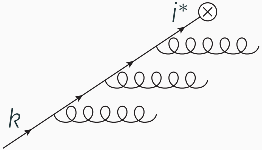
Matching relation is an operator relation (OPE)

$$B_{ik} = \sum_j I_{ij} \otimes f_{jk}$$

$\rightarrow$ Still same matching coefficients I_{ij}

$\rightarrow$ Task becomes: Calculate B_{ik} in perturbative QCD

Which partonic beam functions B_{ik} are there?



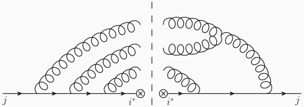
$$B_{ik} = \sum_j I_{ij} \otimes f_{jk}$$

i^*: Off-shell parton entering hard process

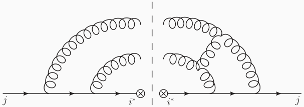
k: External parton (external states)

- Independent flavour combinations:
  $(ik) \in \{qq, qg, q\bar{q}, gg, gq\}$
- Subdivide according to number of loops:

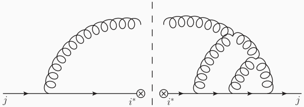
RRR



RRV



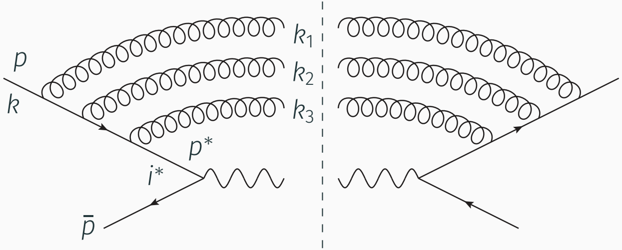
RVV



How to calculate partonic beam functions B_{ik}

Beam functions have operator definitions in SCET, but [Ritzmann, Waalewijn '14] observed: Matching coefficients can be calculated from collinear limits of QCD amplitudes

$$B_{ik}^{\text{bare}} \sim \int \prod_{i=1}^{n_R} \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}} \delta_+(k_i^2)\, \delta\left(2p \cdot k_{1\dots n_R} - \frac{t}{z}\right) \delta\left(\frac{2\bar{p} \cdot k_{1\dots n_R}}{s} - (1-z)\right) \frac{\hat{C}_p |M(p,\bar{p},\{k_i\})|^2}{|M_0(zp,\bar{p})|^2}$$



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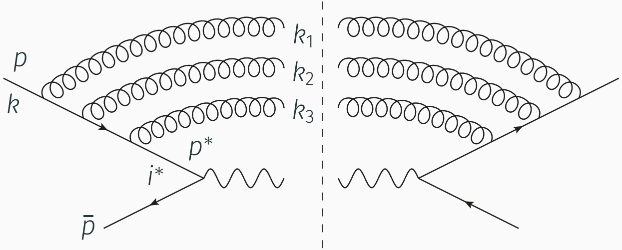

- Construct collinear limits (i.e. splitting functions) à la [Catani, Grazzini ’99]

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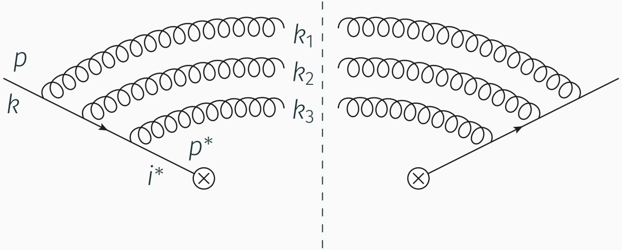

- Construct collinear limits (i.e. splitting functions) à la [Catani, Grazzini '99]
  - Consider only one leg

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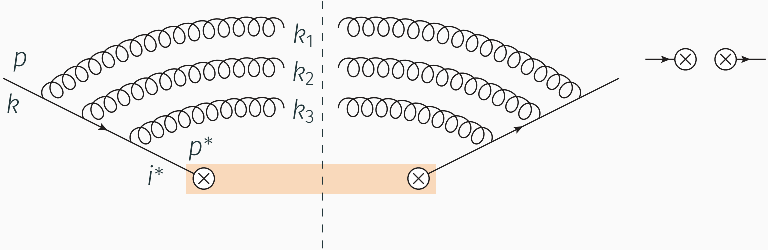

$$\to \frac{\bar{\not{p}}}{4N_c p^* \cdot \bar{p}}$$

- Construct collinear limits (i.e. splitting functions) à la [Catani, Grazzini ’99]
    - Consider only one leg
    - Replace hard process by suitable projector

How to calculate partonic beam functions B_{ik}

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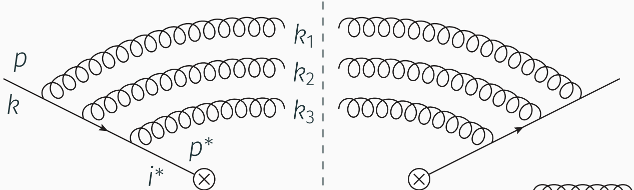

$$\rightarrow \frac{\bar{\not{p}}}{4N_c p^*\cdot\bar{p}}$$

$$\rightarrow \frac{i\left(-g^{\mu\nu} + \frac{k^\mu\bar{p}^\nu + k^\nu\bar{p}^\mu}{k\cdot\bar{p}}\right)}{k^2 + i0^+}$$

- Construct collinear limits (i.e. splitting functions) à la [Catani, Grazzini '99]
  - Consider only one leg
  - Replace hard process by suitable projector
  - Work in axial gauge

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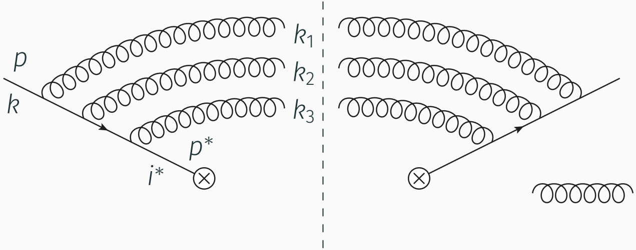

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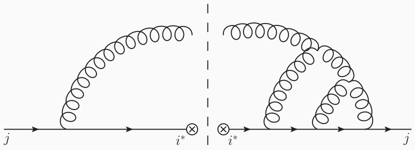
- Integrate over constrained phase space ($k_{1\ldots n_R} = k_1 + \cdots + k_{n_R}$)
  - Implement δ distributions via reverse unitarity

$$\delta(x) = \frac{1}{2\pi i}\left(\frac{1}{x - i0} - \frac{1}{x + i0}\right)$$

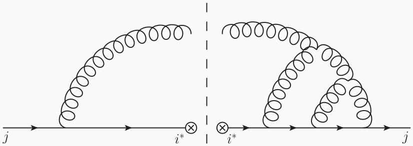
Calculation details: RVV



$$B_{ik}^{\text{bare,RVV}} \sim \int [\mathrm{d}^d k]\, \delta\!\left(2p\cdot k - \frac{t}{z}\right) \delta\!\left(\frac{2\bar{p}\cdot k}{s} - (1-z)\right) \frac{2}{s} \left(-\frac{s}{\mu^2}\right)^{-2\varepsilon} \frac{2\,\mathrm{Re}[P^{(2)}_{k\to i^*}(z)]}{z}$$

- Calculation in axial gauge with two virtual loops is *not fun*™

Calculation details: RVV



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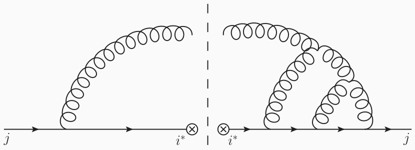
```
arnd@ithil:~/repos/beamfun$ █
```

Calculation details: RVV



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- Calculation in axial gauge with two virtual loops is *not fun*™
- Fortunately, splitting functions $P^{(2)}_{k\to i^*}(z)$ are gauge independent and two-loop results are available in the literature [Duhr, Gehrmann, Jaquier '14]
- Splitting functions are expressed in terms of harmonic polylogarithms
- Integral over single-emission phase space is trivial

Calculation details: RRR and RRV

Standard toolchain:

Diagrams: `QGRAF`

↓

Diagram processing: `FORM` & `Mathematica`

↓

IBPs: `Reduze` & `Kira`

↓

Differential equations

↓

Canonical basis

↓

Boundary constants

↓

Iterated integrals

Calculation details: RRR and RRV

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Iterated integrals

Peculiarities of the calculation

Diagram processing

- Axial gauge propagators and phase space constraints require partial fraction decomposition.
- Due to $\delta(\frac{2}{s}(k_1 + k_2) \cdot \bar{p} - (1 - z))$ we have

$$\frac{1}{(k_1 \cdot \bar{p})(k_2 \cdot \bar{p})} = \frac{2}{s(1-z)} \left[\frac{1}{k_1 \cdot \bar{p}} + \frac{1}{k_2 \cdot \bar{p}} \right]$$

Calculation details: RRR and RRV

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Iterated integrals

Peculiarities of the calculation

Integration-by-parts reduction

- Number of master integrals: 851
- Apply partial fraction relations again to find identities between MI
- Afterwards: Just 473 master integrals

Calculation details: RRR and RRV

Standard toolchain:

Diagrams: `QGRAF`

↓

Diagram processing: `FORM` & `Mathematica`

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Differential equations

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Canonical basis

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↓

Iterated integrals

Peculiarities of the calculation

Differential equations

- 13 letters appear in the differential equation
- Three different square roots appear: $\sqrt{z}\sqrt{4-z}$, $\sqrt{z}\sqrt{4+z}$, $\sqrt{4+z^2}$

Canonical basis

- Due to simultaneous square roots we had to construct parts of the canonical basis by hand

Calculation details: RRR and RRV

Standard toolchain:

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↓

Diagram processing: `FORM` & `Mathematica`

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↓

Differential equations

↓

Canonical basis

↓

Boundary constants

↓

Iterated integrals

Peculiarities of the calculation

Boundary constants

- Fix boundary constants in limit $z \to 1$
- These integrals can be related to master integrals for $gg \to H$ in threshold limit
- We independently recalculated large fraction of boundary constants for N^3LO Higgs production

Calculation details: RRR and RRV

Standard toolchain:

Diagrams: `QGRAF`

↓

Diagram processing: `FORM` & `Mathematica`

↓

IBPs: `Reduze` & `Kira`

↓

Differential equations

↓

Canonical basis

↓

Boundary constants

↓

Iterated integrals

Peculiarities of the calculation

Results in terms of iterated integrals

- Final results have only 5 letters
- Only one square root survives: $\sqrt{z}\sqrt{4-z}$

Results

- There are five independent matching coefficients: $\mathcal{I}_{q_i q_j}$, $\mathcal{I}_{q_i \bar{q}_j}$ $\mathcal{I}_{qg}$, $\mathcal{I}_{gq}$, $\mathcal{I}_{gg}$
  (Recall: $B_i = \sum_j \mathcal{I}_{ij} \otimes f_j$)
- We have computed all of them at N^{3}LO in QCD
  [AB, Melnikov, Rietkerk, Tancredi, Wever '19],
  [Baranowski, AB, Melnikov, Tancredi, Wever '22]

Results

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Checks

✓ Poles cancel after renormalisation and IR subtraction
  $\rightarrow$ Other direction: also cross-check of three-loop splitting functions

✓ Coefficients of plus distributions in z and t agree with predictions from literature
  [Billis, Ebert, Michel, Tackmann ’19]

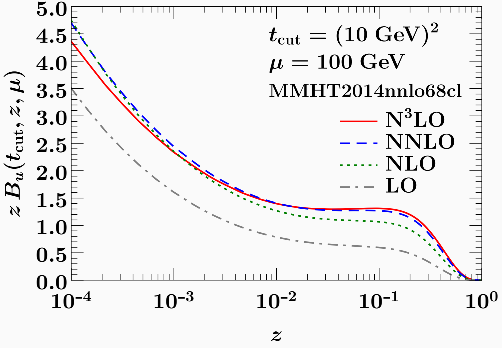
✓ Final result for matching coefficients agrees with independent calculation
  [Ebert, Mistlberger, Vita ’20]

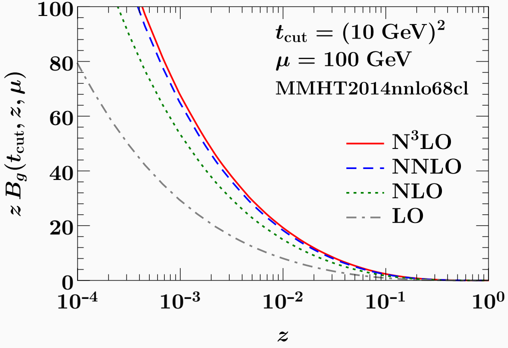
Results: Observations

- Significant simplification of iterated integral alphabet:
  13 letters $\to$ 5 letters
  3 square roots $\to$ 1 square root
- Remaining square root letters appears in very compact and localised structure

$$
\begin{aligned}
&+\frac{21}{8}L_{1,1}-\frac{28}{3}L_{0,0,1}-\frac{82}{9}L_{0,1,0}+\frac{21}{2}L_{0,1,1}-\frac{1}{3}L_{1,0,-1}-\frac{59}{6}L_{1,0,0}\\
&+\frac{64}{9}L_{1,0,1}+\frac{73}{9}L_{1,1,0}-\frac{14}{3}L_{1,1,1}-\frac{22}{9}L_{1,2,1}\Big)\pi^2+\Big(13L_1+34L_{0,1}\\
&+42L_{1,0}-\frac{76}{3}L_{1,1}\Big)\zeta_3+\frac{371}{1080}L_1\pi^4\Big)+\frac{1}{1080}(187\bar{z}+92)\pi^4\\
&-\frac{3\sqrt{1-\bar{z}}(5\bar{z}+14)}{\sqrt{3+\bar{z}}}\Big[L_{r,0,1}+\frac{2}{3}L_{r,1,1}-\frac{\pi^2}{6}L_r\Big]-15\bar{z}\Big[L_{r,r,0,1}\\
&+\frac{2}{3}L_{r,r,1,1}-\frac{\pi^2}{6}L_{r,r}\Big]+6(\bar{z}-2)\Big[L_{1,r,r,0,1}+\frac{2}{3}L_{1,r,r,1,1}-\frac{\pi^2}{6}L_{1,r,r}\Big]\\
&+\frac{\bar{z}^2-2\bar{z}+2}{\bar{z}}\Big(18\Big[L_{0,r,r,0,1}+\frac{2}{3}L_{0,r,r,1,1}-\frac{\pi^2}{6}L_{0,r,r}\Big]\\
&-6\Big[L_{1,r,r,0,1}+\frac{2}{3}L_{1,r,r,1,1}-\frac{\pi^2}{6}L_{1,r,r}\Big]\Big)+\Big(\frac{1}{18}(-\bar{z}^2-22\bar{z}+26)\\
&+\frac{1}{36}(-108\bar{z}+11)L_0+\frac{1}{18}(55\bar{z}-47)L_1-\frac{16}{9}\bar{z}L_{0,0}+\frac{1}{9}(18\bar{z}-1)L_{0,1}\\
&+\frac{1}{12}(55\bar{z}-52)L_{1,0}+\frac{1}{72}(-553\bar{z}+586)L_{1,1}\Big)\pi^2+\Big(\frac{1}{6}(235\bar{z}-18)
\end{aligned}
$$

Results





Plots: [Ebert, Mistlberger, Vita '20]

$$B_i(t_{\rm cut}, z, \mu) = \int_0^{t_{\rm cut}} \mathrm{d}t\, B_i(t, z, \mu)$$

Outlook

Almost all building blocks for N^{3}LO zero-jettiness slicing are available:

$$\mathrm{d}\sigma \sim B \otimes B \otimes S \otimes H \otimes \mathrm{d}\sigma_{\mathrm{LO}}$$
$$\checkmark \quad \checkmark \quad (\checkmark) \quad \checkmark \quad \checkmark$$

Soft function S for zero-jettiness:

- Partial N^{3}LO results available:
  RRR (same hemisphere): [Baranowski, Delto, Melnikov, Wang '22]
  RRV (& RVV): [Chen, Feng, Jia, Liu '22] [Baranowski, Delto, Melnikov, Pikelner, Wang '24]
- Completion (RRR opposite hemisphere) is work in progress
  $\rightarrow$ See Chen-Yu's talk

Beyond zero-jettiness?

- Beam functions are identical for zero- and N-jettiness.
- Soft function depends on N $\rightarrow$ more challenging for $N > 0$